

REGULARITY THEORY FOR SOLUTIONS TO SECOND ORDER ELLIPTIC OPERATORS WITH COMPLEX COEFFICIENTS AND THE L^p DIRICHLET PROBLEM

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ABSTRACT. We establish a new theory of regularity for elliptic complex valued second order equations of the form $\mathcal{L} = \operatorname{div} A(\nabla \cdot)$, when the coefficients of the matrix A satisfy a natural algebraic condition, a strengthened version of a condition known in the literature as L^p -dissipativity. Precisely, the regularity result is a reverse Hölder condition for L^p averages of solutions on interior balls, and serves as a replacement for the De Giorgi - Nash - Moser regularity of solutions to real-valued divergence form elliptic operators. In a series of papers, Cialdea and Maz'ya studied necessary and sufficient conditions for L^p -dissipativity of second order complex coefficient operators and systems. Recently, Carbonaro and Dragičević introduced a condition they termed p -ellipticity, and showed that it had implications for boundedness of certain bilinear operators that arise from complex valued second order differential operators. Their p -ellipticity condition is exactly our strengthened version of L^p -dissipativity. The regularity results of the present paper are applied to solve L^p Dirichlet problems for $\mathcal{L} = \operatorname{div} A(\nabla \cdot) + B \cdot \nabla$ when A and B satisfy a natural and familiar Carleson measure condition. We show solvability of the L^p Dirichlet boundary value problem for p in the range where A is p -elliptic.

1. INTRODUCTION

In this paper, we establish an interior regularity of solutions to second order divergence form complex coefficient operators $\mathcal{L} = \operatorname{div} A(x) \nabla + B(x) \cdot \nabla$ under certain natural algebraic conditions on the matrix A and a natural minimal scaling condition on B , without any additional smoothness of the coefficients. If the coefficients of A and B are real, the algebraic conditions on A are precisely uniform ellipticity.

The improvements in regularity of solutions, as expressed as (1.10) and (1.11) of Theorem 1.1 below, can be used as a substitute for the De Giorgi-Nash-Moser regularity theory for real divergence form elliptic equations. In the latter case, we know that when A, B are real valued and A is elliptic, the regularity theory for solutions gives that $u \in C^\alpha(B)$; this need not hold for solutions to complex coefficient operators.

In the case of complex coefficients, the usual ellipticity assumption is that there exist constants $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda |\xi|^2 \leq \operatorname{Re} \sum_{i,j=0}^{n-1} A_{ij}(x) \xi_i \overline{\xi_j} = \operatorname{Re} \langle A(x) \xi, \xi \rangle \quad \text{and} \quad |\langle A \xi, \eta \rangle| \leq \Lambda |\xi| |\eta| \quad (1.1)$$

for all $\xi, \eta \in \mathbb{C}^n$ and a.e. $x \in \Omega$. Here, we shall assume a slightly stronger form of ellipticity, a strengthening of the concept of L^p dissipativity as defined in a series of papers by Cialdea and Maz'ya [7–9], which in turn was motivated by understanding when semigroups generated by second order elliptic operators are contractive in L^p .

In particular, it had long been known that scalar second order elliptic operators with real coefficients generate contractive semigroups in L^p for all $1 \leq p \leq \infty$.

In [7], the following condition was shown to be sufficient for L^p dissipativity:

$$\frac{4}{pp'} \langle \mathcal{R}e A(x) \xi, \xi \rangle + \langle (\mathcal{R}e A(x) \eta, \eta) + 2 \langle p^{-1} \mathcal{I}m A(x) - p'^{-1} \mathcal{I}m A^t(x) \rangle \xi, \eta \rangle \geq 0, \quad (1.2)$$

for all $\xi, \eta \in \mathbb{R}^n$.

We shall assume a slightly stronger condition, and use a change of variables $\xi = \frac{\sqrt{pp'}}{2} \lambda$ to write it as follows: For some $\varepsilon > 0$ and all $\lambda, \eta \in \mathbb{R}^n$

$$\langle \mathcal{R}e A \lambda, \lambda \rangle + \langle \mathcal{R}e A \eta, \eta \rangle + \left\langle \left(\sqrt{\frac{p'}{p}} \mathcal{I}m A - \sqrt{\frac{p}{p'}} \mathcal{I}m A^t \right) \lambda, \eta \right\rangle \geq \varepsilon (|\lambda|^2 + |\eta|^2). \quad (1.3)$$

We have recently found this same condition in the paper [6], where the authors introduce it in the following form. For $p > 1$ define the \mathbb{R} -linear map $\mathcal{J}_p : \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$\mathcal{J}_p(\alpha + i\beta) = \frac{\alpha}{p} + i \frac{\beta}{p'}$$

where $p' = p/(p-1)$ and $\alpha, \beta \in \mathbb{R}^n$. They define the matrix A to be p -elliptic if for a.e. $x \in \Omega$

$$\mathcal{R}e \langle A(x) \xi, \mathcal{J}_p \xi \rangle \geq \lambda_p |\xi|^2, \quad \forall \xi \in \mathbb{C}^n \quad (1.4)$$

for some $\lambda_p > 0$.

Henceforth, A will be called p -elliptic if it satisfies (1.4) and the upper bound

$$|\langle A(x) \xi, \eta \rangle| \leq \Lambda |\xi| |\eta|, \quad \forall \xi, \eta \in \mathbb{C}^n. \quad (1.5)$$

A short calculation shows that (1.4) and (1.3) are equivalent.

We shall adopt the notation of [6], and recall their observation that this strengthened ellipticity condition is equivalent to $\Delta_p(A) > 0$ where

$$\Delta_p(A) = \operatorname{ess\,inf}_{x \in \Omega} \min_{|\xi|=1} \mathcal{R}e \langle A(x) \xi, \mathcal{J}_p \xi \rangle. \quad (1.6)$$

Observe that when $p = 2$ this is just the usual ellipticity condition (1.1). The p -ellipticity condition $\Delta_p(A) > 0$ can be restated in a different form

$$|1 - 2/p| < \mu(A), \quad (1.7)$$

where

$$\mu(A) = \operatorname{ess\,inf}_{(x, \xi) \in \Omega \times \mathbb{C}^n \setminus \{0\}} \mathcal{R}e \frac{\langle A(x) \xi, \xi \rangle}{|\langle A(x) \xi, \bar{\xi} \rangle|}. \quad (1.8)$$

(c.f. Proposition 5.14 of [6]). The advantage of writing the inequality in this form is that it separates A from p . It also immediately implies that if a matrix A is elliptic (i.e., (1.1) holds) then there exists $p_0 \in [1, 2)$ such that A is p -elliptic if and only if $p \in (p_0, p'_0)$, where $p_0 = 2/(1 + \mu(A))$. Moreover, $p_0 = 1$ if and only if the matrix A is real and the quantity $\mu(A)$ is trivially bounded by $\mu(A) \geq \lambda/\Lambda$ giving a trivial upper bound on the value of p_0 .

Our first main result concerns solutions to $\mathcal{L} = \operatorname{div} A(x) \nabla + B(x) \cdot \nabla$ in a domain $\Omega \subset \mathbb{R}^n$.

Theorem 1.1. *Suppose that $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ is the weak solution to the operator $\mathcal{L}u := \operatorname{div}A(x)\nabla u + B(x) \cdot \nabla u = 0$ in Ω . Let $p_0 = \inf\{p > 1 : A \text{ is } p\text{-elliptic}\}$, and suppose that B has measurable coefficients $B_i \in L_{loc}^\infty(\Omega)$ satisfying the condition*

$$|B_i(x)| \leq K(\delta(x))^{-1}, \quad \forall x \in \Omega \quad (1.9)$$

where the constant K is uniform, and $\delta(x)$ denotes the distance of x to the boundary of Ω . Then we have the following improvement in the regularity of u . For any $B_{4r}(x) \subset \Omega$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left(\int_{B_r(x)} |u|^p dy \right)^{1/p} \leq C_\varepsilon \left(\int_{B_{2r}(x)} |u|^q dy \right)^{1/q} + \varepsilon \left(\int_{B_{2r}(x)} |u|^2 dy \right)^{1/2} \quad (1.10)$$

for all $p, q \in (p_0, \frac{p_0 n}{n-2})$. (Here $p'_0 = p_0/(p_0 - 1)$ and when $n = 2$ one can take $p, q \in (p_0, \infty)$.) The constant in the estimate depends on the dimension, the p -ellipticity constants, Λ , K and $\varepsilon > 0$ but not on $x \in \Omega$, $r > 0$ or u . Moreover, for all $p \in (p_0, p'_0)$ and any $\varepsilon > 0$

$$r^2 \int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \leq C_\varepsilon \int_{B_{2r}(x)} |u(y)|^p dy + \varepsilon \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{p/2}, \quad (1.11)$$

where the constant again depend only on the dimension, p , Λ , K and $\varepsilon > 0$. In particular, $|u|^{(p-2)/2}u$ belongs to $W_{loc}^{1,2}(\Omega; \mathbb{C})$.

Remark. Clearly, if $q \geq 2$ in (1.10) and if $p \geq 2$ in (1.11) one can take $\varepsilon = 0$ as the L^2 average of u can be controlled by the first term on the right hand side of each of the two inequalities.

In general, one can not expect a larger range of p in the reverse Hölder condition of (1.10). In [23], Mayboroda gives a counterexample to (1.10) when $q = 2$ and for any $p > \frac{2n}{n-2}$ under the assumption of (1.1) (which is the same as 2-ellipticity).

We apply this regularity result to the question of solvability of L^p Dirichlet problem for elliptic operators of this type. This part of the paper is motivated by the known results concerning boundary value problems for second order elliptic equations in divergence form, when the coefficients are **real** and satisfy a certain natural, minimal smoothness condition (refer [13, 14, 22]). The literature on solvability of boundary value problems for complex coefficient operators in \mathbb{R}^n is limited, except when the matrix A is of block form. For block matrices A in $\mathcal{L} = \operatorname{div}A(x)\nabla$, there are numerous results on L^p -solvability of the Dirichlet, regularity and Neumann problems, starting with the solution of the Kato problem, where the coefficients of the block matrix are also assumed to be independent of the transverse variable (this assumption is usually referred in literature as “ t -independent”, in our notation it is the x_0 variable). See [5] and [18] and the references therein. For matrices not of block form, there are solvability results in various special cases assuming that the solutions satisfy De Giorgi - Nash - Moser estimates. See [1] and [17] for example. The latter paper is also concerned with operators that are t -independent. Finally, there are perturbation results in a variety of special cases, such as [3] and [2]; the first paper shows that solvability in L^2 implies solvability in L^p for p near 2, and the second paper has L^2 -solvability results for small L^∞ perturbations of real elliptic operators when the complex matrix is t -independent.

Our solvability result for operators of the form $\mathcal{L} = \operatorname{div} A(x) \nabla + B(x) \cdot \nabla$ can be applied on a domains above a Lipschitz graph in \mathbb{R}^n . We do not assume “ t -independence”. Instead, we assume the coefficients A and B satisfy a natural Carleson condition that has appeared in the literature so far only for real elliptic operators. ([22], [13], and [14]). The Carleson condition on A , (1.12) below, holds uniformly on Lipschitz subdomains, and is thus a natural condition in the context of chord-arc domains as well. The paper [19] connects geometric information about the boundary of the domain to information about the elliptic measure of operators that satisfy some closely related conditions.

The second main theorem of the paper establishes the solvability of $\mathcal{L}u = 0$ with L^p Dirichlet boundary data for variable coefficient complex coefficient operators satisfying these Carleson conditions on coefficients.

Theorem 1.2. *Let $1 < p < \infty$, and let Ω be the upper half-space $\mathbb{R}_+^n = \{(x_0, x') : x_0 > 0 \text{ and } x' \in \mathbb{R}^{n-1}\}$. Consider the operator*

$$\mathcal{L}u = \partial_i (A_{ij}(x) \partial_j u) + B_i(x) \partial_i u$$

and assume that the matrix A is p -elliptic with constants λ_p, Λ and $\mathcal{I}m A_{0j} = 0$ for all $1 \leq j \leq n-1$ and $A_{00} = 1$. Assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} [|\nabla A(x)|^2 + |B(x)|^2] \delta(x) dx \quad (1.12)$$

is a Carleson measure in Ω . Let us also denote

$$d\mu'(x) = \sup_{B_{\delta(x)/2}(x)} \left[\sum_j |\partial_0 A_{0j}|^2 + \left| \sum_j \partial_j A_{0j} \right|^2 + |B(x)|^2 \right] \delta(x) dx. \quad (1.13)$$

Then there exist $K = K(\lambda_p, \Lambda, \|\mu\|_C, n, p) > 0$ and $C(\lambda_p, \Lambda, \|\mu\|_C, n, p) > 0$ such that if

$$\|\mu'\|_C < K \quad (1.14)$$

then the L^p -Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = f & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \tilde{N}_{p,a}(u) \in L^p(\partial\Omega), \end{cases} \quad (1.15)$$

is solvable and the estimate

$$\|\tilde{N}_{p,a}(u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{C})} \quad (1.16)$$

holds for all energy solutions u with datum f .

In the statement of this theorem, we’ve used some notation that will be defined in subsequent sections. We will also recall there the concept of Carleson measure, discuss the notions of L^p solvability and energy solutions and define \tilde{N}_p which is a variant of the nontangential maximal function defined using L^p averages of the solution u . By Theorem 1.1, instead of L^p averages we could use L^q averages for q in the range $(p_0, \frac{p_0 n}{n-2})$ to obtain the same result. Also, see section 4 for a detailed discussion of some further assumptions we make to prove Theorem 1.2 on Lipschitz domains.

Classically, the L^p boundedness of the nontangential maximal function of a solution (in our case, estimate (1.16) above) gives nontangential convergence of the

solution to its boundary values. Since the nontangential maximal function of our complex-valued solution will require smoothing by averaging, we will also get a nontangential convergence result, but stated for averages of solutions. This convergence of averages is a consequence of solvability in L^p and is not connected with the assumptions on the coefficients of the equation. For this reason, that very general result is given in an appendix at the end.

We now state, as a further corollary of the second main theorem, a result for matrices A in “block form”. This corollary uses the fact that the assumption that the Carleson measure norm is small is needed only on the last row and last column of the matrix A . This latter observation was pointed out to us by S. Mayboroda - see also [10]. Hence, we have the following.

Corollary 1.3. *Suppose the operator \mathcal{L} on \mathbb{R}_+^n has the form*

$$\mathcal{L}u = \partial_0^2 u + \sum_{i,j=1}^{n-1} \partial_i (A_{ij} \partial_j u)$$

where the matrix A has coefficients satisfying the Carleson condition (1.12).

Then for all $1 < p < \infty$ for which A is p -elliptic, the L^p -Dirichlet problem (4.1) is solvable for \mathcal{L} and the estimate

$$\|\tilde{N}_{p,a} u\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{C})} \quad (1.17)$$

holds for all energy solutions u with datum f .

Finally, we remark that similar results for symmetric elliptic systems have recently been obtained by M. Dindoš, S. Hwang, and M. Mitrea ([12]), namely the L^2 solvability of symmetric elliptic systems under the same Carleson measure assumptions of Theorem 1.2. In particular, the stopping time argument in section 6 is due to [12]. Note that, due to the assumption that the elliptic systems are symmetric, their L^2 solvability result does not apply to our setting of complex coefficient equations.

2. L^p DISSIPATIVITY, p -ELLIPTICITY AND REGULARITY RESULTS

The concept of L^p dissipativity was defined in a series of papers by Cialdea and Maz'ya [7–9] and was motivated by the effort to characterize when semigroups generated by second order elliptic operators are contractive in L^p . In particular, it has long been known that scalar second order elliptic operators with real coefficients generate contractive semigroups in L^p for all $1 \leq p \leq \infty$. The case of operators generating L^∞ -contractive semigroups is studied in [4].

Following [7] let $\mathcal{L}(u, v)$ be the the sesquilinear form

$$\mathcal{L}(u, v) = \int_{\Omega} \langle A(x) \nabla u, \nabla v \rangle dx,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on \mathbb{C}^n . Clearly, $\mathcal{L}(u, v)$ is well defined for $(u, v) \in C_0^1(\Omega) \times C_0^1(\Omega)$ consisting of complex value functions having compact support in Ω with continuous first derivative.

Definition 2.1. (Cialdea-Maz'ya) *Let $1 < p < \infty$. The form \mathcal{L} is called L^p dissipative if for all $u \in C_0^1(\Omega)$*

$$\operatorname{Re} \mathcal{L}(u, |u|^{p-2} u) \geq 0, \quad \text{if } p \geq 2, \quad (2.1)$$

$$\operatorname{Re} \mathcal{L}(u |u|^{p'-2}, u) \geq 0, \quad \text{if } 1 < p < 2. \quad (2.2)$$

Theorem 2.2. [7, Theorem 1, Corollary 4 and Corollary 6] *A sufficient condition for the form \mathcal{L} to be L^p dissipative is that*

$$\frac{4}{pp'} \langle \Re A(x)\xi, \xi \rangle + \langle (\Re A(x)\eta, \eta) \rangle + 2\langle p^{-1} \Im A(x) - p'^{-1} \Im A^t(x) \rangle \xi, \eta \rangle \geq 0, \quad (2.3)$$

for all $\xi, \eta \in \mathbb{R}^n$. If in addition the matrix $\Im A$ be symmetric, i.e., $\Im A = \Im A^t$ then this condition is also necessary and is equivalent to

$$|p - 2| |\langle \Im A(x)\xi, \xi \rangle| \leq 2\sqrt{p-1} \langle \Re A(x)\xi, \xi \rangle, \quad \forall \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega. \quad (2.4)$$

(2.4) must hold even in the non-symmetric case, but then this condition might not be sufficient.

In particular, if we set

$$\tilde{\mu} = \operatorname{ess\,inf}_{(x,\xi) \in \mathcal{M}} \frac{\langle \Re A(x)\xi, \xi \rangle}{|\langle \Im A(x)\xi, \xi \rangle|} \quad (2.5)$$

where \mathcal{M} is the set of (ξ, x) , $x \in \mathbb{R}^n$, $x \in \Omega$ such that $\langle \Im A(x)\xi, \xi \rangle \neq 0$. If $\Im A = 0$ for any $x \in \Omega$ then \mathcal{L} is L^p dissipative for all $p > 1$. If $\Im A$ is symmetric but does not vanish identically on Ω then \mathcal{L} is L^p dissipative if and only if

$$2 + 2\tilde{\mu} \left(\tilde{\mu} - \sqrt{\tilde{\mu}^2 + 1} \right) \leq p \leq 2 + 2\tilde{\mu} \left(\tilde{\mu} + \sqrt{\tilde{\mu}^2 + 1} \right). \quad (2.6)$$

For our purposes (2.3) is not sufficient as in particular this condition does not imply ellipticity when $p = 2$. To guarantee ellipticity, a stronger lower bound is needed, namely that the left hand side of (2.3) is greater than $\varepsilon(|\xi|^2 + |\eta|^2)$. This yields the condition (1.3), which when $p = 2$ is just the usual ellipticity for complex coefficients.

As we have observed in the introduction, (1.3) can be simply written as $\Delta_p(A) > 0$, where (1.6) defines $\Delta_p(A)$. (This was introduced in [6]) and is in turn equivalent to $|1 - 2/p| < \mu(A)$. Hence the following holds.

Theorem 2.3. *Let $A \in L^\infty(\Omega)$ be a matrix that is uniformly elliptic: for some $\lambda, \Lambda > 0$ and almost every $x \in \Omega$ we have*

$$\lambda |\xi|^2 \leq \Re \sum_{i,j=0}^{n-1} A_{ij}(x) \xi_i \bar{\xi}_j \quad \text{and} \quad |\langle A\xi, \eta \rangle| \leq \Lambda |\xi| |\eta| \quad (2.7)$$

for all $\xi, \eta \in \mathbb{C}^n$. Then there exists $p_0 \in [1, 2)$ (with $p_0 = 1$ if and only if $\Im A = 0$) such that the matrix A is p -elliptic if and only if $p \in (p_0, p'_0)$. That is $\Delta_p(A) > 0$, equivalently

$$\langle \Re A \lambda, \lambda \rangle + \langle \Re A \eta, \eta \rangle + \left\langle \left(\sqrt{\frac{p'}{p}} \Im A - \sqrt{\frac{p}{p'}} \Im A^t \right) \lambda, \eta \right\rangle \geq \varepsilon(p) (|\lambda|^2 + |\eta|^2), \quad (2.8)$$

for some $\varepsilon(p) > 0$ and all $\lambda, \eta \in \mathbb{R}^n$. Here $p_0 = \frac{2}{1+\mu(A)}$ where

$$\mu(A) = \operatorname{ess\,inf}_{(x,\xi) \in \mathcal{M}} \Re \frac{\langle A(x)\xi, \xi \rangle}{|\langle A(x)\xi, \xi \rangle|} \geq \frac{\lambda}{\Lambda}, \quad (2.9)$$

where \mathcal{M} is the set (ξ, x) , $x \in \mathbb{C}^n$, $x \in \Omega$ such that $\langle A(x)\xi, \bar{\xi} \rangle \neq 0$. If in addition the matrix A has symmetric imaginary part ($\Im A = \Im A^t$) this further simplifies

to

$$p_0 = 2 + 2\tilde{\mu} \left(\mu - \sqrt{\tilde{\mu}^2 + 1} \right), \quad (2.10)$$

where

$$\tilde{\mu} = \operatorname{ess\,inf}_{(x,\xi) \in \widetilde{\mathcal{M}}} \frac{\langle \mathcal{R}e A(x)\xi, \xi \rangle}{|\langle \mathcal{I}m A(x)\xi, \xi \rangle|} \quad (2.11)$$

and $\widetilde{\mathcal{M}}$ is the set of (ξ, x) , $\xi \in \mathbb{R}^n$, $x \in \Omega$ such that $\langle \mathcal{I}m A(x)\xi, \xi \rangle \neq 0$.

We apply the concept of p -ellipticity to prove the following result.

Theorem 2.4. *Assume that the matrix A is p -elliptic. Then there exists $\lambda'_p = \lambda'_p(\lambda, \Lambda, \lambda_p) > 0$ such that for any nonnegative, bounded and measurable function χ and any u such that $|u|^{(p-2)/2}u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$, we have*

$$\mathcal{R}e \int_{\Omega} \langle A(x)\nabla u, \nabla(|u|^{p-2}u) \rangle \chi(x) dx \geq \lambda'_p \int_{\Omega} |u|^{p-2} |\nabla u|^2 \chi(x) dx. \quad (2.12)$$

Proof. Since A is p -elliptic (2.8) holds. Changing the variables $\lambda = \frac{2}{\sqrt{pp'}}\xi$ and obtain

$$\begin{aligned} & \frac{4}{pp'} \langle \mathcal{R}e A \xi, \xi \rangle + \langle \mathcal{R}e A \eta, \eta \rangle + \\ & + 2 \langle p^{-1} \mathcal{I}m A - p'^{-1} \mathcal{I}m A^t \rangle \xi, \eta \geq \varepsilon' (|\xi|^2 + |\eta|^2). \end{aligned} \quad (2.13)$$

Consider now $v \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ and write $|v|^{-1}\bar{v}\nabla v$ as $X + iY$, that is

$$X = \mathcal{R}e(|v|^{-1}\bar{v}\nabla v) \quad \text{and} \quad Y = \mathcal{I}m(|v|^{-1}\bar{v}\nabla v).$$

Let $\chi(x)$ be a nonnegative bounded and measurable function on Ω . By using X in place of ξ and Y in place of η , multiplying by $\chi(x)$ and integrating over Ω one obtains from (2.13) (similar to [7, Corollary 4])

$$\begin{aligned} & \mathcal{R}e \int_{\Omega} \left[\langle A\nabla v, \nabla v \rangle - (1 - 2/p) \langle (A - A^*)\nabla(|v|), |v|^{-1}\bar{v}\nabla v \rangle \right. \\ & \left. - (1 - 2/p)^2 \langle A\nabla(|v|), \nabla(|v|) \rangle \right] \chi(x) dx \geq \varepsilon' \int_{\Omega} |\nabla v|^2 \chi(x) dx \end{aligned} \quad (2.14)$$

Now as in (2.9) of [7] one considers $v = |u|^{p/2-1}u$ and

$$g_{\varepsilon} = (|v|^2 + \varepsilon^2)^{1/2}, \quad u_{\varepsilon} = g_{\varepsilon}^{2/p-1}v,$$

for u, u_{ε} . Using our assumption we see that $|u|^{(p-2)/2}u, |u_{\varepsilon}|^{(p-2)/2}u_{\varepsilon} \in W_{loc}^{1,2}(\Omega, \mathbb{C}^n)$. We let $\varepsilon \rightarrow 0+$. When $p > 2$, we obtain, using Lebesgue dominated convergence,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0+} \mathcal{R}e \int_{\Omega} \langle A\nabla u_{\varepsilon}, \nabla(|u_{\varepsilon}|^{p-2}u_{\varepsilon}) \rangle \chi(x) dx = \\ & \mathcal{R}e \int_{\Omega} \left[\langle A\nabla v, \nabla v \rangle - (1 - 2/p) \langle (A - A^*)\nabla(|v|), |v|^{-1}\bar{v}\nabla v \rangle \right. \\ & \left. - (1 - 2/p)^2 \langle A\nabla(|v|), \nabla(|v|) \rangle \right] \chi(x) dx \geq \varepsilon' \int_{\Omega} |\nabla(|u|^{p/2-1}u)|^2 \chi(x) dx. \end{aligned} \quad (2.15)$$

and then the first term of (2.15) is just

$$\mathcal{R}e \int_{\Omega} \langle A(x)\nabla u, \nabla(|u|^{p-2}u) \rangle \chi(x) dx. \quad (2.16)$$

When $1 < p < 2$, we use a duality argument based on an observation in [7]. Set $w = |u|^{p-2}u$, so that $u = |w|^{p'-2}w$. The fact that $|u|^{(p-2)/2}u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ implies that $|w|^{(p'-2)/2}w \in W_{loc}^{1,2}(\Omega; \mathbb{C})$. Then,

$$\Re \int_{\Omega} \langle A(x) \nabla u, \nabla(|u|^{p-2}u) \rangle \chi(x) dx = \Re \int_{\Omega} \langle A^*(x) \nabla w, \nabla(|w|^{p'-2}w) \rangle \chi(x) dx,$$

and we have that A^* is p' -elliptic when A is p -elliptic. Therefore, we have reduced the regime of the first case considered, and the same limiting argument yields

$$\Re \int_{\Omega} \langle A^*(x) \nabla w, \nabla(|w|^{p'-2}w) \rangle \chi(x) dx \geq \lambda'_{p'} \int_{\Omega} |w|^{p'-2} |\nabla w|^2 \chi(x) dx. \quad (2.17)$$

We conclude the argument for (2.12) in the case $p < 2$ by observing that

$$\int_{\Omega} |w|^{p'-2} |\nabla w|^2 \chi(x) dx \approx \int_{\Omega} |\nabla(|w|^{p'/2-1}w)|^2 \chi(x) dx = \int_{\Omega} |\nabla(|u|^{p/2-1}u)|^2 \chi(x) dx.$$

□

The following observation will be used frequently.

Lemma 2.5. *For all $p > 1$, and for all x for which $u(x) \neq 0$*

$$|\nabla(|u(x)|^{p/2-1}u(x))|^2 \approx |u(x)|^{p-2} |\nabla u(x)|^2.$$

Proof. For $k = 0, 1, \dots, n-1$ we have

$$\partial_k(|u|^{(p-2)/2}u) = |u|^{(p-4)/2} \left[|u| \partial_k u + \frac{p-2}{2} u \partial_k(|u|) \right].$$

Multiplying by its complex conjugate then yields

$$\begin{aligned} \left| \partial_k(|u|^{(p-2)/2}u) \right|^2 &= |u|^{p-4} \left[|u|^2 |\partial_k u|^2 + (p-2) |u| \partial_k(|u|) \Re \langle u, \partial_k u \rangle \right. \\ &\quad \left. + \left(\frac{p-2}{2} \right)^2 |u|^2 (\partial_k(|u|))^2 \right]. \end{aligned}$$

As in the proof of [8, Lemma 2] we have $\partial_k(|u|) = |u|^{-1} \Re \langle u, \partial_k u \rangle$ which yields

$$\left| \partial_k(|u|^{(p-2)/2}u) \right|^2 = |u|^{p-4} \left[|u|^2 |\partial_k u|^2 + \left[(p-2) + \left(\frac{p-2}{2} \right)^2 \right] |\Re \langle u, \partial_k u \rangle|^2 \right].$$

Summing over all k gives us

$$\left| \nabla(|u|^{(p-2)/2}u) \right|^2 = |u|^{p-4} \left[|u|^2 |\nabla u|^2 + \left[\left(\frac{p}{2} \right)^2 - 1 \right] \sum_k |\Re \langle u, \partial_k u \rangle|^2 \right].$$

Since by Cauchy Schwartz

$$\sum_k |\Re \langle u, \partial_k u \rangle|^2 \leq |u|^2 |\nabla u|^2,$$

then clearly

$$\left| \nabla(|u|^{(p-2)/2}u) \right|^2 \leq \left(1 + \left| \left(\frac{p}{2} \right)^2 - 1 \right| \right) |u|^{p-2} |\nabla u|^2,$$

and for $p > 0$

$$\min \left\{ 1, \left(\frac{p}{2} \right)^2 \right\} |u|^{p-2} |\nabla u|^2 \leq \left| \nabla(|u|^{(p-2)/2}u) \right|^2.$$

Hence the claim holds. □

We now turn to the main lemmas required to prove Theorem 1.1.

Lemma 2.6. *Let the matrix A be p -elliptic for $p \geq 2$ and let B have coefficients satisfying condition (1.9) of Theorem 1.1. Suppose that u is a $W_{loc}^{1,2}(\Omega; \mathbb{C})$ solution to \mathcal{L} in Ω . Then, for any ball $B_r(x)$ with $r < \delta(x)/4$,*

$$\int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \lesssim r^{-2} \int_{B_{2r}(x)} |u(y)|^p dy \quad (2.18)$$

and

$$\left(\int_{B_r(x)} |u(y)|^q dy \right)^{1/q} \lesssim \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{1/2} \quad (2.19)$$

for all $q \in (2, \frac{np}{n-2}]$ when $n > 2$, and where the implied constants depend only p -ellipticity and K of (1.9). When $n = 2$, q can be any number in $(2, \infty)$. In particular, $|u|^{(p-2)/2}u$ belongs to $W_{loc}^{1,2}(\Omega; \mathbb{C})$.

Proof. We begin by assuming that both A and B are smooth. Thus the solution u to $\mathcal{L}(u) = 0$ will be smooth in the interior of Ω ([24]) and we have (2.12) at our disposal for any p such that A is p -elliptic. We prove (2.18) first, and then (2.19) by a bootstrap argument so that no constants appearing depend on the smoothness of the coefficients of A, B .

Let $B_r(x)$ be ball in the interior of Ω with $r < \delta(x)/4$. Let φ be a smooth cutoff function, with $\varphi = 1$ on $B_r(x)$ and vanishing outside of $B_{2r}(x)$. Set $v = u\varphi$. Then,

$$\mathcal{L}v = u\mathcal{L}\varphi + A\nabla u \cdot \nabla \varphi + A^* \nabla u \cdot \nabla \varphi. \quad (2.20)$$

Multiply both sides of (2.20) by $|v|^{p-2}\bar{v}$ and integrate by parts to obtain

$$\begin{aligned} \int \nabla(|v|^{p-2}\bar{v}) \cdot A\nabla v dy &= \int (|v|^{p-2}\bar{v})B \cdot \nabla v dy + \int \nabla(|v|^{p-2}\bar{v}u) \cdot A\nabla \varphi dy \\ &\quad - \int |v|^{p-2}\bar{v}u B \cdot \nabla \varphi dy - \int |v|^{p-2}\bar{v}A\nabla u \cdot \nabla \varphi dy \\ &\quad - \int |v|^{p-2}\bar{v}A^* \nabla u \cdot \nabla \varphi dy \end{aligned} \quad (2.21)$$

The real part of the left hand side of (2.21) is bounded from below by $\lambda_p \int |v|^{p-2} |\nabla v|^2 dy$ by p -ellipticity and (2.12). The first of the five terms on the right hand side above has the bound

$$\left| \int (|v|^{p-2}\bar{v}) \cdot B\nabla v dy \right| \lesssim K r^{-1} \left(\int |v|^{p-2} |\nabla v|^2 dy \right)^{1/2} \left(\int |v|^p dy \right)^{1/2} \quad (2.22)$$

where K is as in (1.9). The third term has the bound

$$\left| \int |v|^{p-2}\bar{v}u B \cdot \nabla \varphi dy \right| \lesssim K r^{-2} \int_{B_{2r}(x)} |u|^p dy. \quad (2.23)$$

In terms four and five, we use the fact that $\bar{v}\nabla u = \bar{u}\nabla v - |u|^2 \nabla \varphi$ to bound each of these integrals by a constant multiple of

$$\int |v|^{p-2} |u| |\nabla \varphi| |\nabla v| + |v|^{p-2} |u|^2 |\nabla \varphi|^2 dy \quad (2.24)$$

with constant depending on the ellipticity parameter Λ . By Cauchy-Schwarz, this integral is bounded by terms like those in (2.22) and (2.23). After distributing the gradient, the second term has the following bound

$$\int |\nabla(|v|^{p-2}\bar{v})||u||\nabla\varphi| + |v|^{p-2}|\bar{v}\nabla u||\nabla\varphi| dy \quad (2.25)$$

Using Lemma 2.5 to compute the derivative in the first expression, the first term in the above integral is bounded by

$$\int |v|^{p-2}|\nabla v||u||\nabla\varphi| dy \quad (2.26)$$

and the second expression, using the same trick as in (2.24) is bounded by

$$\int |v|^{p-2}|\nabla v||u||\nabla\varphi| + |v|^{p-2}|u|^2|\nabla\varphi|^2 dy \quad (2.27)$$

Taking the real part of both sides of (2.21) and combining all these estimates, and noting that the constants do not depend on the smoothness of the A or B gives (2.18) for this u . Observe, in addition, that the Sobolev embedding gives

$$\left(\int_{B_r(x)} |u|^{\tilde{p}} dy\right)^{1/\tilde{p}} \lesssim \left(\int_{B_{2r}(x)} |v|^{\tilde{p}} dy\right)^{1/\tilde{p}} \lesssim \left(r^2 \int_{B_{2r}(x)} |\nabla(|v|^{p/2-1}v)|^2 dy\right)^{1/p} \quad (2.28)$$

where $\tilde{p} = \frac{pn}{n-2}$.

First we use (2.28) and the argument for (2.18) (see (2.21) and the subsequent estimates) to obtain a reverse Hölder inequality for u . That is,

$$\left(\int_{B_r(x)} |u|^{\tilde{p}} dy\right)^{1/\tilde{p}} \lesssim \left(\int_{B_{\alpha r}(x)} |u|^p dy\right)^{1/p} \quad (2.29)$$

Note that at this moment we have only proven (2.29) for $\alpha = 2$; however, by adjusting the cutoff function φ , the entire argument can be done with balls $B_{\alpha r}(x)$, for any $\alpha > 1$, and with a new constant which will depend on α . Iterating (2.29) k times gives

$$\left(\int_{B_r(x)} |u|^{p_k} dy\right)^{1/p_k} \lesssim \left(\int_{B_{\alpha^k r}(x)} |u|^2 dy\right)^{1/2} \quad (2.30)$$

for $p_k = 2(\frac{n}{n-2})^k$, as long as $p_{k-1} < p$.

It remains to remove the assumption of smoothness of the coefficients of the operator, noting that the constants appearing in (2.18) and (2.19) depend only on the p -ellipticity and K and not on any smoothness parameter. Suppose that A satisfies the condition of p -ellipticity and B satisfies (1.9), and let A_j and B_j be smooth approximations, converging a.e. to A , B , respectively. Let u_j be the solution to $\mathcal{L}_j u_j = 0$ with the same boundary data as u . We claim that the arguments of section 7 of [21] can be used to show that $u_j \rightarrow u$ strongly in $W^{1,2}$ on compact subsets of Ω . We note that the arguments of [21] did not consider the convergence of lower order terms, but this will follow in the same way from the strong convergence of u_j to u in L^q for $q = 2n/(n-2)$. From the fact that (2.19) holds for u_j , and the strong convergence of u_j to u in $L^2(B_{2r})$, we see that the L^q averages of u_j are uniformly bounded. Passing to a subsequence if necessary, we have weak convergence of the u_j in L^q , and that weak limit must be u . This

implies that the L^q average of u is bounded as well, i.e., we have (2.19) for u . Now set $w_j = |u_j(x)|^{p/2-1}u_j(x)$ and $w = |u(x)|^{p/2-1}u(x)$. From (2.18) for u_j and (2.19) for u , we have that w_j is uniformly bounded in $W_{loc}^{1,2}(\Omega; \mathbb{C})$. This gives weak convergence of w_j to a limit, which again must be w . The weak convergence gives the uniform bound on $|\nabla w|^2$, and thus (2.18) for u . \square

Lemma 2.7. *Let the matrix A be p -elliptic for $p < 2$ and let B have coefficients satisfying condition (1.9) of Theorem 1.1. Suppose that u is a $W_{loc}^{1,2}(\Omega; \mathbb{C})$ solution to \mathcal{L} in Ω . Then, for any ball $B_r(x)$ with $r < \delta(x)/4$ and any $\varepsilon > 0$*

$$r^2 \int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \leq C_\varepsilon \int_{B_{2r}(x)} |u(y)|^p dy + \varepsilon \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{p/2} \quad (2.31)$$

and

$$\left(\int_{B_r(x)} |u(y)|^2 dy \right)^{1/2} \leq C_\varepsilon \left(\int_{B_{2r}(x)} |u(y)|^p dy \right)^{1/p} + \varepsilon \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{1/2} \quad (2.32)$$

where the constants depend only p -ellipticity, K of (1.9) and $\varepsilon > 0$. In particular, $|u|^{(p-2)/2}u$ belongs to $W_{loc}^{1,2}(\Omega; \mathbb{C})$.

Proof. We assume that A and B are smooth, and remove this assumption later. We can therefore, as before, assume that we have a smooth solution to \mathcal{L} . The proof of this lemma proceeds in a similar fashion as in (2.19), but without introducing the cutoff function φ . This will lead to presence of the two extra terms (with ε) on the right hand sides of (2.31) and (2.32) which were not needed in (2.18) and (2.19). Unfortunately, if we attempt to proceed as in Lemma 2.6 with $v = u\varphi$ we will run into trouble estimating terms like (2.27) as $|v|^{p-2} \leq |u|^{p-2}$ when $p \geq 2$, but in our case $p < 2$ this does not hold and we run into problems near the boundary of supp φ where φ can be arbitrary small.

The fact that we do not introduce the cutoff function φ does mean we have fewer terms to take care of but we do have to worry about one additional boundary integral and more importantly we also have to introduce an additional approximation to account for the fact that $p - 2$ is negative. To that end, define a cutoff function ρ_δ as follows.

$$\rho_\delta(s) = \begin{cases} \delta^{\frac{p-2}{2}}, & \text{if } 0 \leq s \leq \delta \\ s^{\frac{p-2}{2}}, & \text{if } s > \delta. \end{cases} \quad (2.33)$$

Following the proof of Lemma 2.7, we multiply both sides of the equation $\mathcal{L}u = 0$ by $\rho_\delta(|u|)^2 \bar{u}$ and integrate:

$$\begin{aligned} \int_{B_r(x)} \nabla(\rho_\delta^2(|u|)\bar{u}) \cdot A \nabla u \, dy &= \int_{B_r(x)} (\rho_\delta^2(|u|)\bar{u}) B \cdot \nabla u \, dy \\ &\quad + \int_{\partial B_r(x)} (\rho_\delta^2(|u|)\bar{u}) \nu \cdot A \nabla u \, d\sigma(y). \end{aligned} \quad (2.34)$$

Here ν is the outer unit normal which for the ball is just $\frac{y-x}{r}$.

The left hand side of (2.34) splits into terms:

$$\begin{aligned} \int_{B_r(x)} \nabla(\rho_\delta^2(|u|)\bar{u}) \cdot A \nabla u \, dy &= \delta^{p-2} \int_{B_r(x) \setminus E_\delta} \langle A \nabla u, \nabla u \rangle \, dy \\ &\quad + \int_{E_\delta \cap B_r(x)} A \nabla u \cdot \nabla(|u|^{p-2}\bar{u}) \, dy \end{aligned} \quad (2.35)$$

where $E_\delta = \{|u| > \delta\}$. Applying Theorem 2.4 on the open set $E_\delta \cap B_r(x)$, we have that

$$\mathcal{R}e \int_{E_\delta \cap B_r(x)} A \nabla u \cdot \nabla(|u|^{p-2}\bar{u}) \, dy \geq \lambda_p \int_{E_\delta \cap B_r(x)} |u|^{p-2} |\nabla u|^2 \, dy \quad (2.36)$$

Ultimately we will let $\delta \rightarrow 0$, and we will show that the integrals involving $B_r(x) \setminus E_\delta$ will tend to zero. The main tool will be the following fact, established in [15] for smooth functions u , namely

$$\delta^r \int_{B_r(x) \setminus E_\delta} |\nabla u|^2 \, dy \rightarrow 0 \quad (2.37)$$

for all $r > -1$.

We first take care of the boundary integral in (2.34). Observe that (2.34)-(2.36) hold not only on the ball $B_r(x)$ but on any enlarged ball $B_{\alpha r}(x)$ for $1 \leq \alpha \leq 3/2$. Hence if we write (2.34) for each such α and then average over the interval $[1, 3/2]$ the last term of (2.34) will turn into a solid integral over the set $B_{3r/2}(x) \setminus B_r(x)$.

This and (2.35)-(2.36) then yields

$$\begin{aligned} \lambda_p \int_{E_\delta \cap B_r(x)} |u|^{p-2} |\nabla u|^2 \, dy &\leq \sup_{\alpha \in [1, 3/2]} \left| \int_{B_{\alpha r}(x)} (\rho_\delta^2(|u|)\bar{u}) B \cdot \nabla u \, dy \right| \\ &\quad + \left| r^{-1} \int_{B_{3r/2}(x) \setminus B_r(x)} (\rho_\delta^2(|u|)\bar{u}) \frac{(y-x) \cdot A \nabla u}{|y-x|} \, dy \right| + o(1), \end{aligned} \quad (2.38)$$

where $o(1)$ contains the integral over the complement of E_δ , which tend to zero as $\delta \rightarrow 0$.

Each of the two terms on the right hand side of (2.38) will split into two integrals, one on $B_{3r/2}(x) \setminus E_\delta$ and one on $E_\delta \cap B_{3r/2}(x)$. Clearly,

$$\begin{aligned} \left| \int_{B_{\alpha r}(x)} (\rho_\delta^2(|u|)\bar{u}) B \cdot \nabla u \, dy \right| &\lesssim r^{-1} \int_{E_\delta \cap B_{3r/2}(x)} |u|^{p-1} |\nabla u| \, dy \\ &\quad + \delta^{p-1} r^{-1} \int_{B_{3r/2}(x) \setminus E_\delta} |\nabla u| \, dy, \end{aligned} \quad (2.39)$$

and

$$\begin{aligned} \left| r^{-1} \int_{B_{3r/2}(x) \setminus B_r(x)} (\rho_\delta^2(|u|)\bar{u}) \frac{(y-x) \cdot A \nabla u}{|y-x|} \, dy \right| &\lesssim r^{-1} \int_{E_\delta \cap B_{3r/2}(x)} |u|^{p-1} |\nabla u| \, dy \\ &\quad + \delta^{p-1} r^{-1} \int_{B_{3r/2}(x) \setminus E_\delta} |\nabla u| \, dy. \end{aligned} \quad (2.40)$$

where the implied constants in (2.39) and (2.40) depend on K and Λ respectively. The last terms in both inequalities behave like $C\delta^{p-1}$, and will tend to zero as

$\delta \rightarrow 0$ and hence can be written as $o(1)$ terms. By Hölder inequality we have for the first terms on the right hand side of (2.39)-(2.40):

$$\begin{aligned} r^{-1} \int_{E_\delta \cap B_{3r/2}(x)} |u|^{p-1} |\nabla u| dy &\leq r^{-1} \left(\int_{B_{3r/2}(x)} |u|^p dy \right)^{(p-1)/p} \left(\int_{B_{3r/2}(x)} |\nabla u|^p dy \right)^{1/p} \\ &\leq C_\varepsilon r^{-2} \int_{B_{3r/2}(x)} |u|^p dy + \varepsilon r^{p-2} \int_{B_{3r/2}(x)} |\nabla u|^p dy, \end{aligned} \quad (2.41)$$

for any $\varepsilon > 0$.

Putting all terms together therefore yields

$$\lambda_p r^2 \int_{E_\delta} |v|^{p-2} |\nabla v|^2 dy \leq C_\varepsilon \int_{B_{3r/2}(x)} |u|^p dy + \varepsilon r^p \int_{B_{3r/2}(x)} |\nabla u|^p dy + o(1), \quad (2.42)$$

where $o(1)$ contains all integrals over the complement of E_δ , which tend to zero as $\delta \rightarrow 0$. The constant C_ε depends on Λ and K (and ε) but not on the smoothness of A and B . Let δ tend to zero and obtain

$$r^2 \int_{\{y: u(y) \neq 0\} \cap B_r(x)} |u|^{p-2} |\nabla u|^2 dy \leq C'_\varepsilon \int_{B_{2r}(x)} |u|^p dy + \varepsilon r^p \int_{B_{3r/2}(x)} |\nabla u|^p dy. \quad (2.43)$$

Recalling the convention that $|u|^{p-2} |\nabla u|^2$ is taken to be zero whenever $\nabla u = 0$, the integral on the left hand side of (2.43) can be taken on the set $B_r(x) \setminus \{y : u(y) = 0, \nabla u(y) \neq 0\}$. However the measure of $\{y : u(y) = 0, \nabla u(y) \neq 0\}$ is zero and so we can conclude that (after introducing averages)

$$r^2 \oint_{B_r(x)} |u|^{p-2} |\nabla u|^2 dy \leq C''_\varepsilon \oint_{B_{2r}(x)} |u|^p dy + \varepsilon r^p \oint_{B_{3r/2}(x)} |\nabla u|^p dy \quad (2.44)$$

holds. The final step is to use Hölder inequality and Caccioppoli inequality (which is just (2.18) of Lemma 2.6 when $p = 2$) for the last term of (2.44). We have

$$r^p \oint_{B_{3r/2}(x)} |\nabla u|^p dy \lesssim \left(r^2 \oint_{B_{3r/2}(x)} |\nabla u|^2 dy \right)^{p/2} \lesssim \left(\oint_{B_{2r}(x)} |u|^2 dy \right)^{p/2}. \quad (2.45)$$

Finally, (2.44) and (2.45) combined yields (2.31).

The argument for (2.32) is similar to that for the case $p > 2$. First, we have the analog of (2.29).

$$\left(\oint_{B_r(x)} |u|^{\bar{p}} dy \right)^{1/\bar{p}} \leq C_\varepsilon \left(\oint_{B_{\alpha r}(x)} |u|^p dy \right)^{1/p} + \varepsilon \left(\oint_{B_{\alpha r}(x)} |u|^2 dy \right)^{1/2} \quad (2.46)$$

The iteration scheme starts with a $p < 2$ and stops when we reach a p_k greater than or equal to 2. Each iteration will give us additional term of the form

$$\left(\oint_{B_{\alpha^i r}(x)} |u|^2 dy \right)^{1/2}$$

on the right hand side for $i = 1, 2, \dots, k$ multiplied by a constant that can be as small as required, since at each step of the iteration ε in (2.46) can be chosen independently of the previous choices.

Finally, since the constants in (2.31) and (2.32) do not depend on the smoothness of the coefficients of A and B , arguments very similar to those in Lemma 2.6 allow us to pass to the limit and obtain these inequalities for solutions to \mathcal{L} under the assumptions of the Lemma. \square

The reverse Hölder inequalities of Lemmas 2.6 and 2.7 prove Theorem 1.1.

3. CARLESON MEASURES, NONTANGENTIAL MAXIMAL FUNCTIONS AND p -ADAPTED SQUARE FUNCTIONS

3.1. Nontangential maximal and square functions. On a domain of the form

$$\Omega = \{(x_0, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > \phi(x')\}, \quad (3.1)$$

where $\phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a Lipschitz function with Lipschitz constant given by $L := \|\nabla \phi\|_{L^\infty(\mathbb{R}^{n-1})}$, define for each point $x = (x_0, x') \in \Omega$

$$\delta(x) := x_0 - \phi(x') \approx \text{dist}(x, \partial\Omega). \quad (3.2)$$

In other words, $\delta(x)$ is comparable to the distance of the point x from the boundary of Ω .

Definition 3.1. *A cone of aperture $a > 1$ is a non-tangential approach region to the point $Q \in \partial\Omega$ defined as*

$$\Gamma_a(Q) = \{x \in \Omega : |x - Q| < a\delta(x)\}. \quad (3.3)$$

We require $1 < a < \frac{L}{\sqrt{1+L^2}}$, otherwise the aperture of the cone is too large. But when $\Omega = \mathbb{R}_+^n$ all parameters $a > 1$ may be considered. Sometimes it is necessary to truncate $\Gamma(Q)$ at height h , in which case we write

$$\Gamma_a^h(Q) := \Gamma_a(Q) \cap \{x \in \Omega : \delta(x) \leq h\}. \quad (3.4)$$

Definition 3.2. *For $\Omega \subset \mathbb{R}^n$ as above, the square function of some $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by*

$$S_a(u)(Q) := \left(\int_{\Gamma_a(Q)} |\nabla u(x)|^2 \delta(x)^{2-n} dx \right)^{1/2} \quad (3.5)$$

and, for each $h > 0$, its truncated version is given by

$$S_a^h(u)(Q) := \left(\int_{\Gamma_a^h(Q)} |\nabla u(x)|^2 \delta(x)^{2-n} dx \right)^{1/2}. \quad (3.6)$$

A simple application of Fubini's theorem gives

$$\|S_a(u)\|_{L^2(\partial\Omega)}^2 \approx \int_{\Omega} |\nabla u(x)|^2 \delta(x) dx. \quad (3.7)$$

In [DPP], a “ p -adapted” square function was introduced in order to solve Dirichlet problems in the range $1 < p < 2$. We shall use this method, and a similar p -adapted square function, but for both the ranges $p \geq 2$ and $p < 2$. In the following definition, when $p < 2$ we use the convention that the expression $|\nabla u(x)|^2 |u(x)|^{p-2}$ is zero whenever $\nabla u(x)$ vanishes.

Definition 3.3. For $\Omega \subset \mathbb{R}^n$, the p -adapted square function of $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$S_{p,a}(u)(Q) := \left(\int_{\Gamma_a(Q)} |\nabla u(x)|^2 |u(x)|^{p-2} \delta(x)^{2-n} dx \right)^{1/2} \quad (3.8)$$

and, for each $h > 0$, its truncated version is given by

$$S_{p,a}^h(u)(Q) := \left(\int_{\Gamma_a^h(Q)} |\nabla u(x)|^2 |u(x)|^{p-2} \delta(x)^{2-n} dx \right)^{1/2}. \quad (3.9)$$

We have shown that the expressions of the form $|\nabla u(x)|^2 |u(x)|^{p-2}$, when u is a solution of $\mathcal{L}u = 0$ are locally integrable and hence the definition of $S_p(u)$ makes sense for such p .

Definition 3.4. For $\Omega \subset \mathbb{R}^n$ as above, and for a continuous $u : \Omega \rightarrow \mathbb{C}$, the nontangential maximal function (h -truncated nontangential maximal function) of u at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$, is defined by

$$N_a(u)(Q) := \sup_{x \in \Gamma_a(Q)} |u(x)| \quad \text{and} \quad N_a^h(u)(Q) := \sup_{x \in \Gamma_a^h(Q)} |u(x)|. \quad (3.10)$$

Moreover, we shall also consider a related version of the above nontangential maximal function. This is denoted by $\tilde{N}_{p,a}$ and is defined using L^p averages over balls in the domain Ω . Specifically, given $u \in L_{loc}^p(\Omega; \mathbb{C})$ we set

$$\tilde{N}_{p,a}(u)(Q) := \sup_{x \in \Gamma_a(Q)} w(x) \quad \text{and} \quad \tilde{N}_{p,a}^h(u)(Q) := \sup_{x \in \Gamma_a^h(Q)} w(x) \quad (3.11)$$

for each $Q \in \partial\Omega$ and $h > 0$ where, at each $x \in \Omega$,

$$w(x) := \left(\oint_{B_{\delta(x)/2}(x)} |u(z)|^p dz \right)^{1/p}. \quad (3.12)$$

Above and elsewhere, a barred integral indicates an averaging operation. Observe that, given $u \in L_{loc}^p(\Omega; \mathbb{C})$, the function w associated with u as in (3.12) is continuous and $\tilde{N}_{p,a}(u) = N_a(w)$ everywhere on $\partial\Omega$.

The L^2 -averaged nontangential maximal function was introduced in [21] in connection with the Neuman and regularity problem value problems. In the context of p -ellipticity, Theorem 1.1 shows that there is no difference between L^2 averages and L^p averages, a fact which we record in the following proposition.

Proposition 3.5. Suppose that u is a $W_{loc}^{1,2}(\Omega; \mathbb{C})$ solution to $\mathcal{L} = \text{div}A(x)\nabla + B(x) \cdot \nabla$ in Ω , where the matrix A is assume to be p -elliptic for $p \in (p_0, p'_0)$, and B satisfies condition (1.9). Then, for every $Q \in \partial\Omega$

$$\tilde{N}_{p,a}(u)(Q) \lesssim \tilde{N}_{q,a'}(u)(Q) \quad \forall p, q \in \left(p_0, \frac{p'_0 n}{n-2} \right) \quad (3.13)$$

when the aperture parameters of the cones satisfy $a < a'$ and $p \leq q$ or $q \geq 2$.

When $a < a'$, $q < 2$, $p > q$ and $p, q \in \left(p_0, \frac{p'_0 n}{n-2} \right)$ we have a weaker estimate

$$\tilde{N}_{p,a}(u)(Q) \lesssim \tilde{N}_{q,a'}(u)(Q) + \varepsilon \tilde{N}_{2,a'}(u)(Q), \quad (3.14)$$

for all $\varepsilon > 0$.

Finally, for any aperture parameters a, a' in the appropriate range, and for all p, q in the same range as (3.13),

$$\|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)} \approx \|\tilde{N}_{p,a'}(u)\|_{L^r(\partial\Omega)} \quad (3.15)$$

for all $r > 0$.

Proof. Clearly, (3.13) and (3.14) follows from (1.10) (as we have noted earlier the integral over B_{2r} in (1.10) can be replaced by $B_{\alpha r}$ for any $\alpha > 1$ and hence the cones $\Gamma_{a'}(Q)$ used on the right hand side of (3.13) and (3.14) can be just little bit larger than the cone $\Gamma_a(Q)$ used in definition of $\tilde{N}_{p,a}(u)$ on the left hand side of (3.13) and (3.14)).

Looking at the equivalence in the L^r norm, consider for the moment still the case when $a' > a$. When $p < q$ or $q \geq 2$

$$\|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} \quad (3.16)$$

follows from (3.13). Otherwise by (3.14) applied to $p = 2$ yields

$$\|\tilde{N}_{2,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} + \varepsilon \|\tilde{N}_{2,a'}(u)\|_{L^r(\partial\Omega)}. \quad (3.17)$$

The norm equivalence of the nontangential maximal functions $N_{p,a}$ and $\tilde{N}_{p,a'}$ for different aperture parameters a, a' requires only a classical real-variable argument using the level sets $\{\tilde{N}_{p,a}(u) > \lambda\}$ and $\{\tilde{N}_{p,a'}(u) > \lambda\}$. Hence always

$$\|\tilde{N}_{p,a'}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)}. \quad (3.18)$$

Combining this with (3.19) implies that

$$\|\tilde{N}_{2,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} + \varepsilon \|\tilde{N}_{2,a}(u)\|_{L^r(\partial\Omega)}, \quad (3.19)$$

which by choosing $\varepsilon > 0$ sufficiently small then gives

$$\|\tilde{N}_{2,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)}. \quad (3.20)$$

Combining this with (3.16) for any $p \in \left(p_0, \frac{p_0 n}{n-2}\right)$ and $a < a'' < a'$ we then have

$$\|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{2,a''}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} \quad (3.21)$$

as desired. Hence (3.16) holds for any p, q in our range as long as $a' > a$.

The reverse inequality to (3.16) can be obtained using (3.18) by choosing $a'' < a$. This gives

$$\|\tilde{N}_{q,a'}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{q,a''}(u)\|_{L^r(\partial\Omega)}$$

and then since $a'' < a$ by (3.16)

$$\|\tilde{N}_{q,a''}(u)\|_{L^r(\partial\Omega)} \lesssim \|\tilde{N}_{p,a}(u)\|_{L^r(\partial\Omega)}.$$

□

3.2. Carleson measures. We begin by recalling the definition of a Carleson measure in a domain Ω as in (3.1). For $P \in \mathbb{R}^n$, define the ball centered at P with the radius $r > 0$ as

$$B_r(P) := \{x \in \mathbb{R}^n : |x - P| < r\}. \quad (3.22)$$

Next, given $Q \in \partial\Omega$, by $\Delta = \Delta_r(Q)$ we denote the surface ball $\partial\Omega \cap B_r(Q)$. The Carleson region $T(\Delta_r)$ is then defined by

$$T(\Delta_r) := \Omega \cap B_r(Q). \quad (3.23)$$

Definition 3.6. A Borel measure μ in Ω is said to be Carleson if there exists a constant $C \in (0, \infty)$ such that for all $Q \in \partial\Omega$ and $r > 0$

$$\mu(T(\Delta_r)) \leq C\sigma(\Delta_r), \quad (3.24)$$

where σ is the surface measure on $\partial\Omega$. The best possible constant C in the above estimate is called the Carleson norm and is denoted by $\|\mu\|_C$.

In all that follows we now assume that the coefficients of the matrix A and B of the elliptic operator $\mathcal{L} = \operatorname{div} A(x) \nabla + B(x) \cdot \nabla$ satisfies the following natural conditions. First, we assume that the entries A_{ij} of A are in $\operatorname{Lip}_{loc}(\Omega)$ and the entries of B are $L^\infty_{loc}(\Omega)$. Second, we assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} [|\nabla A|^2 + |B|^2] \delta(x) dx \quad (3.25)$$

is a Carleson measure in Ω . Sometimes, and for certain coefficients of A , we will assume that their Carleson norm $\|\mu\|_C$ is sufficiently small. Crucially we have the following result.

Theorem 3.7. Suppose that μ the Carleson measure described in (3.25). Then there exists a finite constant $C = C(L, a) > 0$ such that for every $u \in L^p_{loc}(\Omega; \mathbb{C})$ one has

$$\int_{\Omega} |u(x)|^p d\mu(x) \leq C \|\mu\|_C \int_{\partial\Omega} \left(\tilde{N}_{p,a}(u) \right)^p d\sigma. \quad (3.26)$$

3.3. Pullback Transformation. The Carleson measure conditions, (3.25), on the coefficients of \mathcal{L} are compatible with a useful change of variables described in this subsection.

For a domain Ω as in (3.1), consider a mapping $\rho : \mathbb{R}_+^n \rightarrow \Omega$ appearing in works of Dahlberg, Nečas, Kenig-Stein and others, defined by

$$\rho(x_0, x') := (x_0 + P_{\gamma x_0} * \phi(x'), x'), \quad \forall (x_0, x') \in \mathbb{R}_+^n, \quad (3.27)$$

for some positive constant γ . Here P is a nonnegative function $P \in C_0^\infty(\mathbb{R}^{n-1})$ and, for each $\lambda > 0$,

$$P_\lambda(x') := \lambda^{-n+1} P(x'/\lambda), \quad \forall x' \in \mathbb{R}^{n-1}. \quad (3.28)$$

Finally, $P_\lambda * \phi(x')$ is the convolution

$$P_\lambda * \phi(x') := \int_{\mathbb{R}^{n-1}} P_\lambda(x' - y') \phi(y') dy'. \quad (3.29)$$

Observe that ρ extends up to the boundary of \mathbb{R}_+^n and maps one-to-one from $\partial\mathbb{R}_+^n$ onto $\partial\Omega$. Also for sufficiently small $\gamma \lesssim L$ the map ρ is a bijection from $\overline{\mathbb{R}_+^n}$ onto $\overline{\Omega}$ and, hence, invertible.

For $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ that solves $\mathcal{L}u = 0$ in Ω with Dirichlet datum f consider $v := u \circ \rho$ and $\tilde{f} := f \circ \rho$. The change of variables via the map ρ just described implies that $v \in W_{loc}^{1,2}(\mathbb{R}_+^n; \mathbb{C})$ solves a new elliptic PDE of the form

$$0 = \operatorname{div}(\tilde{A}(x) \nabla v) + \tilde{B}(x) \cdot \nabla v, \quad (3.30)$$

with boundary datum \tilde{f} on $\partial\mathbb{R}_+^n$. Hence, solving a boundary value problem for u in Ω is equivalent to solving a related boundary value problem for v in \mathbb{R}_+^n . Crucially, if the coefficients of the original system are such that (3.25) is a Carleson measure, then the coefficients of \tilde{A} and \tilde{B} satisfy an analogous Carleson condition in the upper-half space. If, in addition, the Carleson norm of (3.25) is small and L (the

Lipschitz constant for the domain Ω) is also small, then the Carleson norm for the new coefficients \tilde{A} and \tilde{B} will be correspondingly small. Hence the map ρ allows us to assume that the domain is $\Omega = \mathbb{R}_+^n$.

Moreover, this transformation also preserves p -ellipticity.

4. THE L^p -DIRICHLET PROBLEM

When an operator \mathcal{L} is as in Theorem 1.2 is uniformly elliptic in the sense of (1.1), the Lax-Milgram lemma can be applied and guarantees the existence of weak solutions. That is, given any $f \in B_{1/2}^{2,2}(\partial\Omega; \mathbb{C})$, the space of traces of functions in $W^{1,2}(\Omega; \mathbb{C})$, there exists a unique $u \in W^{1,2}(\Omega; \mathbb{C})$ such that $\mathcal{L}u = 0$ in Ω and $\text{Tr } u = f$ on $\partial\Omega$. We call these solutions “energy solutions” and use them to define the notion of solvability of the L^p Dirichlet problem.

Definition 4.1. *Let Ω be the Lipschitz domain introduced in (3.1) and fix an integrability exponent $p \in (1, \infty)$. Also, fix an aperture parameter $a > 1$. Consider the following Dirichlet problem for a complex valued function $u : \Omega \rightarrow \mathbb{C}$:*

$$\begin{cases} 0 = \partial_i (A_{ij}(x) \partial_j u) + B_i(x) \partial_i u & \text{in } \Omega, \\ u(x) = f(x) & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \tilde{N}_{2,a}(u) \in L^p(\partial\Omega), \end{cases} \quad (4.1)$$

where the usual Einstein summation convention over repeated indices (i, j in this case) is employed.

We say the Dirichlet problem (4.1) is solvable for a given $p \in (1, \infty)$ if there exists a $C = C(\lambda, \Lambda, n, p, \Omega) > 0$ such that for all boundary data $f \in L^p(\partial\Omega; \mathbb{C}) \cap B_{1/2}^{2,2}(\partial\Omega; \mathbb{C})$ the unique “energy solution” satisfies the estimate

$$\|\tilde{N}_{2,a}(u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{C})}. \quad (4.2)$$

Remark. Since the space $B_{1/2}^{2,2}(\partial\Omega; \mathbb{C}) \cap L^p(\partial\Omega; \mathbb{C})$ is dense in $L^p(\partial\Omega; \mathbb{C})$ for each $p \in (1, \infty)$, it follows that there exists a unique continuous extension of the solution operator $f \mapsto u$ to the whole space $L^p(\partial\Omega; \mathbb{C})$, with u such that $\tilde{N}_{2,a}(u) \in L^p(\partial\Omega)$ and the accompanying estimate $\|\tilde{N}_{2,a}(u)\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega; \mathbb{C})}$ being valid.

Moreover, we shall establish in the appendix that under the assumptions of Theorem 1.2 for any $f \in L^p(\partial\Omega; \mathbb{C})$ the corresponding solution u constructed by the continuous extension attains the datum f as its boundary values in the following sense. Consider the average $\tilde{u} : \Omega \rightarrow \mathbb{C}$ defined by

$$\tilde{u}(x) = \oint_{B_{\delta(x)/2}(x)} u(y) dy, \quad \forall x \in \Omega.$$

Then

$$f(Q) = \lim_{x \rightarrow Q, x \in \Gamma(Q)} \tilde{u}(x), \quad \text{for a.e. } Q \in \partial\Omega, \quad (4.3)$$

where the a.e. convergence is taken with respect to the \mathcal{H}^{n-1} Hausdorff measure on $\partial\Omega$.

Let us make some observations that explain the structural assumptions we have made in Theorem 1.2. As we have already stated it suffices to formulate the result in the case $\Omega = \mathbb{R}_+^n$ by using the pull-back map introduced above. Since Theorem 1.2 requires that the coefficients have *small* Carleson norm this puts a restriction

on the size of the Lipschitz constant $L = \|\nabla\phi\|_{L^\infty}$ of the map ϕ that defines the domain Ω in (3.1). The constant L will have also to be small (depending on λ_p , Λ , n and p).

For technical reasons we also need that all coefficients A_{0j} , $j = 0, 1, \dots, n-1$ are real. This can be ensured as follows. When $j > 0$ observe that we have

$$\partial_0([\mathcal{I}m A_{0j}]\partial_i u) = \partial_j([\mathcal{I}m A_{0j}]\partial_0 u) + (\partial_0[\mathcal{I}m A_{0j}])\partial_i u - ([\partial_i \mathcal{I}m A_{0j}])\partial_0 u$$

which allows to move the imaginary part of the coefficient A_{0j} onto the coefficient A_{j0} at the expense of two (harmless) first order terms. This does not work for the coefficient A_{00} . Instead we make the following observation.

Suppose that the measure (3.25) associated to an operator $\mathcal{L} = \partial_i(A_{ij}(x)\partial_j) + B_i(x)\partial_i$ is Carleson. Consider a related operator $\tilde{\mathcal{L}} = \partial_i(\tilde{A}_{ij}(x)\partial_j) + \tilde{B}_i(x)\partial_i$, where $\tilde{A} = \alpha A$ and $\tilde{B} = \alpha B - (\partial_i \alpha)A_{ij}\partial_j$, and $\alpha \in L^\infty(\Omega)$ is a complex valued function such that $|\alpha(x)| \geq \alpha_0 > 0$ and $|\nabla \alpha|^2 x_0$ is a Carleson measure.

Observe that a weak solution u to $\tilde{\mathcal{L}}u = 0$ is also a weak solution to $\mathcal{L}u = 0$ and that the new coefficients of \tilde{A} and \tilde{B} also satisfy a Carleson measure condition as in (3.25), from the assumption on α . We will only require that the coefficient \tilde{A}_{00} is real but we may as well ensure for simplicity that it equals to 1. Clearly, if we choose $\alpha = A_{00}^{-1}$, then the new operator $\tilde{\mathcal{L}}$ will have this property. When A_{00} (and hence α) is real, then \tilde{A} . Similarly, if A is p -elliptic and $\mathcal{I}m A_{00}$ is sufficiently small (depending on the ellipticity constants), then \tilde{A} will also be p -elliptic. However, if $\mathcal{I}m \alpha$ is not small, the p -ellipticity, after multiplication of A by α may not be preserved. Thus, we assume in our main result (Theorem 1.2) the p -ellipticity of the new matrix \tilde{A} which has all coefficients \tilde{A}_{0j} , $j = 0, 1, \dots, n-1$ real, as this is not implied in the general case from the p -ellipticity of the original matrix A .

We are now ready to prove Theorem 1.2.

Proof. As follows from Corollary 5.3

$$\lambda_p' \iint_{\mathbb{R}_+^n} |\nabla u|^2 |u|^{p-2} x_0 dx' dx_0 \leq \int_{\mathbb{R}^{n-1}} |u(0, x')|^p dx' + C \|\mu'\|_C \int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u)]^p dx'. \quad (4.4)$$

Furthermore by using Proposition 6.8 we then have

$$\int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u)]^p dx' \leq C_1 \int_{\mathbb{R}^{n-1}} |u(0, x')|^p dx' + C_2 \|\mu'\|_C \int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u)]^p dx'. \quad (4.5)$$

Here the constants C_1, C_2 depend on p, λ_p, Λ, n and $\|\mu\|_C$. It follows that for

$$\|\mu'\|_C < \frac{1}{2C_2}$$

we have that

$$\int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u)]^p dx' \leq 2C_1 \int_{\mathbb{R}^{n-1}} |u(0, x')|^p dx' \quad (4.6)$$

which proves the estimate (1.16) and hence Theorem 1.2 holds. \square

5. ESTIMATES FOR THE p -ADAPTED SQUARE FUNCTION $S_p(u)$ OF A SOLUTION

In this section we establish a one sided estimate of the p -adapted square function in terms of boundary data and the nontangential maximal function. The approximation of L^p data on the boundary by functions in $B_{1/2}^{2,2}(\partial\Omega; \mathbb{C})$ is considered in the appendix (section 7).

Lemma 5.1. *Let $\Omega = \mathbb{R}_+^n$ and assume u is a weak solution of (4.1) with the Dirichlet boundary datum $f \in B_{1/2}^{2,2}(\partial\Omega; \mathbb{C}) \cap L^p(\partial\Omega; \mathbb{C})$. Assume that A is uniformly elliptic and smooth in \mathbb{R}_+^n with $A_{00} = 1$ and A_{0j} real. Furthermore assume that p satisfies condition (ii) of Theorem 1.2 and that the measure μ defined as in (1.12) is Carleson. Then there exists a constant $C = C(n, N, \lambda, \Lambda, p)$ such that for all $r > 0$*

$$\begin{aligned} & p \frac{\lambda_p}{2} \iint_{[0, r/2] \times \partial\Omega} |u|^{p-2} |\nabla u|^2 x_0 dx' dx_0 + \frac{2}{r} \iint_{[0, r] \times \partial\Omega} |u(x_0, x')|^p dx' dx_0 \\ & \leq \int_{\partial\Omega} |u(0, x')|^p dx' + \int_{\partial\Omega} |u(r, x')|^p dx' + C \|\mu'\|_C \int_{\partial\Omega} \left[\tilde{N}_{p,a}^r(u) \right]^p dx'. \end{aligned} \quad (5.1)$$

Proof. To proceed, fix an arbitrary $y' \in \partial\Omega \equiv \mathbb{R}^{n-1}$ and pick a smooth cutoff function ζ which is x_0 -independent and satisfies

$$\zeta = \begin{cases} 1 & \text{in } B_r(y'), \\ 0 & \text{outside } B_{2r}(y'). \end{cases} \quad (5.2)$$

Moreover, assume that $r|\nabla\zeta| \leq c$ for some positive constant c independent of y' . We begin by considering the integral quantity

$$\mathcal{I} := \mathcal{R}e \iint_{[0, r] \times B_{2r}(y')} A_{ij} \partial_j u \partial_i (|u|^{p-2} \bar{u}) x_0 \zeta dx' dx_0 \quad (5.3)$$

with the usual summation convention understood. With $\chi = x_0 \zeta$ we have by Theorem 2.4 for some $\lambda_p = \lambda_p(\lambda, p) > 0$

$$\mathcal{I} \geq \lambda_p \iint_{[0, r] \times B_{2r}} |u|^{p-2} |\nabla u|^2 x_0 \zeta dx' dx_0, \quad (5.4)$$

where we agree henceforth to abbreviate $B_{2r} := B_{2r}(y')$ whenever convenient.

The idea now is to integrate by parts the formula for \mathcal{I} in order to relocate the ∂_i derivative. This gives

$$\begin{aligned} \mathcal{I} &= \mathcal{R}e \int_{\partial[(0, r) \times B_{2r}]} A_{ij} \partial_j u |u|^{p-2} \bar{u} x_0 \zeta \nu_{x_i} d\sigma \\ &\quad - \mathcal{R}e \iint_{[0, r] \times B_{2r}} \partial_i (A_{ij} \partial_j u) |u|^{p-2} \bar{u} x_0 \zeta dx' dx_0 \\ &\quad - \mathcal{R}e \iint_{[0, r] \times B_{2r}} A_{ij} \partial_j u |u|^{p-2} \bar{u} \partial_i x_0 \zeta dx' dx_0 \\ &\quad - \mathcal{R}e \iint_{[0, r] \times B_{2r}} A_{ij} \partial_j u |u|^{p-2} \bar{u} x_0 \partial_i \zeta dx' dx_0 \\ &=: I + II + III + IV, \end{aligned} \quad (5.5)$$

where ν is the outer unit normal vector to $(0, r) \times B_{2r}$. The boundary term I does not vanish only on the set $\{r\} \times B_{2r}$ and only when $i = 0$. This gives

$$I = \mathcal{R}e \int_{\{r\} \times B_{2r}} A_{0j} \partial_j u |u|^{p-2} \bar{u} x_0 \zeta d\sigma \quad (5.6)$$

As u is a weak solution of $\mathcal{L}u = 0$ in Ω , we use the equation to transform II into

$$II = \mathcal{R}e \iint_{[0,r] \times B_{2r}} B_i (\partial_i u) |u|^{p-2} \bar{u} x_0 \zeta dx' dx_0. \quad (5.7)$$

To further estimate this term we use Hölder's inequality, the Carleson condition for the term B and Theorem 3.7 in order to write

$$\begin{aligned} |II| &\leq \left(\iint_{[0,r] \times B_{2r}} (B_i)^2 |u|^p x_0 \zeta dx' dx_0 \right)^{1/2} \cdot \left(\iint_{[0,r] \times B_{2r}} |u|^{p-2} |\partial_j u|^2 x_0 \zeta dx' dx_0 \right)^{1/2} \\ &\leq C(\lambda, \Lambda, N) \left(\|\mu'\|_C \int_{B_{2r}} \left[\tilde{N}_{p,a}^r(u) \right]^p dx' \right)^{1/2} \cdot \mathcal{I}^{1/2}. \end{aligned} \quad (5.8)$$

As $\partial_i x_0 = 0$ for $i > 0$ the term III is non-vanishing only for $i = 0$. We further split this term by considering the cases when $j = 0$ and $j > 0$. This yields, since $A_{00} = 1$,

$$\begin{aligned} III_{\{j=0\}} &= -\mathcal{R}e \iint_{[0,r] \times B_{2r}} \partial_0 u |u|^{p-2} \bar{u} \zeta dx' dx_0 \\ &= -\frac{1}{p} \iint_{[0,r] \times B_{2r}} \partial_0 (|u|^p) \zeta dx' dx_0 \\ &= -\frac{1}{p} \int_{B_{2r}} |u|^p(r, x') \zeta dx' + \frac{1}{p} \int_{B_{2r}} |u|^p(0, x') \zeta dx' \end{aligned} \quad (5.9)$$

When $j > 0$ we first use the fact that A_{0j} is real and hence the expression $\mathcal{R}e [A_{0j} (\partial_j u) |u|^{p-2} \bar{u}] = p^{-1} A_{0j} \partial_j (|u|^p)$. Then we reintroduce $1 = \partial_0 x_0$ and integrate by parts moving the ∂_0 derivative

$$\begin{aligned} III_{\{j \neq 0\}} &= -\mathcal{R}e \iint_{[0,r] \times B_{2r}} A_{0j} \partial_j u |u|^{p-2} \bar{u} \zeta dx' dx_0 \\ &\quad - p^{-1} \iint_{[0,r] \times B_{2r}} A_{0j} \partial_j (|u|^p) (\partial_0 x_0) \zeta dx' dx_0 \\ &= p^{-1} \int_{B_{2r}} A_{0j} \partial_j (|u|^p)(r, x') r \zeta dx' + p^{-1} \iint_{[0,r] \times B_{2r}} \partial_0 A_{0j} \partial_j (|u|^p) x_0 \zeta dx' dx_0 \\ &\quad + p^{-1} \iint_{[0,r] \times B_{2r}} A_{0j} \partial_{0j}^2 (|u|^p) x_0 \zeta dx' dx_0 \\ &= III_1 + III_2 + III_3. \end{aligned}$$

We note that $III_1 = -I_{\{j \neq 0\}}$.

In the third term III_3 we switch the order of derivatives $\partial_{0j}^2 = \partial_{j0}^2$ and take further integration by parts with respect to ∂_j .

$$\begin{aligned} III_3 &= -p^{-1} \iint_{[0,r] \times B_{2r}} \partial_j A_{0j} \partial_0(|u|^p) x_0 \zeta \, dx' \, dx_0 \\ &\quad - p^{-1} \iint_{[0,r] \times B_{2r}} A_{0j} \partial_0(|u|^p) x_0 (\partial_j \zeta) \, dx' \, dx_0 = III_{31} + III_{32}. \end{aligned}$$

The terms III_2 and III_{31} are of the same type as II we have handled earlier and hence have the same estimate

$$III_2 + III_{31} \leq C(\lambda, \Lambda, N) \left(\|\mu'\|_C \int_{B_{2r}} \left[\tilde{N}_{p,a}^r(u) \right]^p \, dx' \right)^{1/2} \cdot \mathcal{I}^{1/2}$$

We add up all terms we have so far to obtain

$$\begin{aligned} \mathcal{I} &\leq p^{-1} \int_{B_{2r}} \partial_0(|u|^p)(r, x') r \zeta \, dx' \\ &\quad - p^{-1} \int_{B_{2r}} |u|^p(r, x') \zeta \, dx' + p^{-1} \int_{B_{2r}} |u|^p(0, x') \zeta \, dx' \\ &\quad + C(\lambda, \Lambda, n, N) \|\mu'\|_C \int_{B_{2r}} \left[\tilde{N}_{p,a}^r(u) \right]^p (u) \, dx' + \frac{1}{2} \mathcal{I} \\ &\quad + III_{32} + IV. \end{aligned} \tag{5.10}$$

We have used the arithmetic-geometric inequality for expression bounding the term II in (5.8) as well as for similar terms III_2 and III_{31} .

To obtain a global version of (5.10), consider a sequence of disjoint boundary balls $(B_r(y'_k))_{k \in \mathbb{N}}$ such that $\cup_k B_{2r}(y'_k)$ covers $\partial\Omega = \mathbb{R}^{n-1}$ and consider a partition of unity $(\zeta_k)_{k \in \mathbb{N}}$ subordinate to this cover. That is, assume $\sum_k \zeta_k = 1$ on \mathbb{R}^{n-1} and each ζ_k is supported in $B_{2r}(y'_k)$. Write IV_k for each term as the last expression in (5.5) corresponding to $B_{2r} = B_{2r}(y'_k)$. Given that $\sum_k \partial_i \zeta_k = 0$ for each i , by summing (5.10) over all k 's gives $\sum_k IV_k = 0$. The same observation applies to the terms arising in III_{32} . It follows that

$$\begin{aligned} &\frac{\lambda_p}{2} \iint_{[0,r] \times \mathbb{R}^{n-1}} |\nabla u|^2 |u|^{p-2} x_0 \, dx' \, dx_0 \\ &\quad p^{-1} \int_{\mathbb{R}^{n-1}} \partial_0(|u|^p)(r, x') r \zeta \, dx' \\ &\quad - p^{-1} \int_{\mathbb{R}^{n-1}} |u|^p(r, x') \zeta \, dx' + p^{-1} \int_{\mathbb{R}^{n-1}} |u|^p(0, x') \zeta \, dx' \\ &\quad + C \|\mu'\|_C \int_{\mathbb{R}^{n-1}} \left[\tilde{N}_{p,a}^r(u) \right]^p \, dx'. \end{aligned} \tag{5.11}$$

From this, (5.1) follows by integrating (5.11) in r on $[0, r']$ and dividing by r' . \square

Lemma 5.1 has several important corollaries.

Corollary 5.2. *Under the assumptions of Lemma 5.1, for a weak solution u of (4.1), and for any $r > 0$, we have*

$$\iint_{[0, r/2] \times \partial\Omega} |\nabla u|^2 |u|^{p-2} x_0 \, dx' \, dx_0 \leq C_p (1 + \|\mu'\|_C) \int_{\partial\Omega} \left[\tilde{N}_{p,a}^r(u) \right]^p \, dx'. \tag{5.12}$$

That is, $\|S_{p,a}^{r/2}(u)\|_{L^p(\partial\Omega)} \leq C\|\tilde{N}_{p,a}^r(u)\|_{L^p(\partial\Omega)}$ with the intervening constant depending only on λ, Λ, n, p and $\|\mu\|_C$. In particular, letting $r \rightarrow \infty$ yields a version for the global square and nontangential maximal functions, namely

$$\|S_{p,a}(u)\|_{L^p(\partial\Omega)} \leq C\|\tilde{N}_{p,a}(u)\|_{L^p(\partial\Omega)}, \quad (5.13)$$

for all energy solutions u of $\mathcal{L}u = 0$ in Ω .

Corollary 5.3. *Under the assumptions of Lemma 5.1, for any energy solution u of (4.1) we have*

$$\lambda'_p \iint_{\mathbb{R}_+^n} |\nabla u|^2 |u|^{p-2} x_0 dx' dx_0 \leq \int_{\mathbb{R}^{n-1}} |u(0, x')|^p dx' + C\|\mu'\|_C \int_{\mathbb{R}^{n-1}} [\tilde{N}_{p,a}(u)]^p dx'. \quad (5.14)$$

6. BOUNDS FOR THE NONTANGENTIAL MAXIMAL FUNCTION BY THE p -ADAPTED SQUARE FUNCTION

As before, we work on $\Omega = \mathbb{R}_+^n$ and we assume that the matrix A is p -elliptic. Our aim in this section is to establish the converse of the inequality in Corollary 5.2. The approach necessarily differs from the usual argument in the real scalar elliptic case due to the fact that certain estimates, such as interior Hölder regularity of a weak solution, are unavailable for the complex coefficient case. Hence, alternative arguments bypassing such difficulties must be devised. We use here an adaptation of the approach developed for elliptic systems in [12].

The major innovation from [12] is the use of an entire family of Lipschitz graphs on which the nontangential maximal function is large in lieu of a single graph constructed via a stopping time argument. This is necessary as we are using L^p averages of solutions to define the nontangential maximal function and hence the knowledge of certain bounds for a solution on a single graph provides no information about the L^p averages over interior balls.

We are assuming that u is a $W^{1,2}(\mathbb{R}_+^n; \mathbb{C})$ solution to \mathcal{L} . For the function w defined in Ω as in (3.12), and a constant $\nu > 0$, define the set

$$E_{\nu,a} := \{x' \in \partial\Omega : N_a(w)(x') > \nu\} \quad (6.1)$$

(where, as usual, $a > 1$ is a fixed background parameter), and consider the map $h : \partial\Omega \rightarrow \mathbb{R}$ given at each $x' \in \partial\Omega$ by

$$h_{\nu,a}(w)(x') := \inf \left\{ x_0 > 0 : \sup_{z \in \Gamma_a(x_0, x')} w(z) < \nu \right\} \quad (6.2)$$

with the convention that $\inf \emptyset = \infty$. We remark that h differs somewhat from the function that has been used in the argument for scalar equations (cf. [22, pp. 212] and [20]). Observe also that for an energy solution u we have that the function w is continuous on Ω and $w(r, \cdot)$ decays uniformly to 0 as $r \rightarrow \infty$. In fact, as seen from (3.12), we have

$$0 \leq w(r, x') \leq r^{-n/2} \|u\|_{L^2(\mathbb{R}_+^n)} \quad \text{for every } r > 0, \text{ and } x' \in \mathbb{R}^{n-1}. \quad (6.3)$$

The decay estimate (6.3) is clear when w denotes the L^2 -averages of a solution u . The reverse Hölder estimates of Theorem 1.1 give the same decay estimates for the L^p -average.

This guarantees that $h_{\nu,a}(w, x') < \infty$ for all points $x' \in \partial\Omega$. We look at some further properties of this function. The following has been established in [12] for elliptic systems and applies in our situation as well.

Lemma 6.1. *Let u be an energy solution of (4.1), and associate with it the function w as in (3.12). Also, fix two positive numbers ν, a . Then the following properties hold.*

(i) *The function $h_{\nu,a}(w)$ is Lipschitz, with a Lipschitz constant $1/a$. That is,*

$$|h_{\nu,a}(w)(x') - h_{\nu,a}(w)(y')| \leq a^{-1}|x' - y'| \quad (6.4)$$

for all $x', y' \in \partial\Omega$.

(ii) *Given an arbitrary $x' \in E_{\nu,a}$, let $x_0 := h_{\nu,a}(w)(x')$. Then there exists a point $y = (y_0, y') \in \partial\Gamma_a(x_0, x')$ such that $w(y) = \nu$ and $h_{\nu,a}(w)(y') = y_0$.*

Lemma 6.2. *Assume as before that u is an energy solution of (4.1) in $\Omega = \mathbb{R}_+^n$. For any $a > 0$ there exists $b = b(a) > a$ and $\gamma = \gamma(a) > 0$ such that the following holds. Having fixed an arbitrary $\nu > 0$, for each point x' from the set*

$$\{x' : N_a(w)(x') > \nu \text{ and } S_{p,b}(u)(x') \leq \gamma\nu\} \quad (6.5)$$

there exists a boundary ball R with $x' \in 2R$ and such that

$$|w(h_{\nu,a}(w)(z'), z')| > \nu/2^{2/p} \text{ for all } z' \in R. \quad (6.6)$$

Proof. Let $x' \in \partial\Omega$ be such that $N_a(w)(x') > \nu$ and $S_{p,b}(u)(x') \leq \gamma\nu$. Set $x_0 := h_{\nu,a}(w, x')$. From part (ii) in Lemma 6.1 we know that there exists a point $y = (y_0, y') \in \partial\Gamma_a(x_0, x')$ such that $w(y) = \nu$. Let $d := |x' - y'|$ and define $R = \{z' \in \partial\Omega : |z' - y'| < 3ay_0/2\}$. Then $x' \in 2R$ since $d < ay_0$. This choice also guarantees that $h_{\nu,a}(w, z') \in [y_0/3, 5y_0/3]$ by (i) in Lemma 6.1.

To proceed, consider the set

$$\mathcal{O} := \{z = (z_0, z') \in \Omega : z' \in R \text{ and } z_0 \in [y_0/3, 5y_0/3]\}. \quad (6.7)$$

In particular, $y \in \mathcal{O}$. Then all claims in the current lemma are justified as soon as we establish that

$$w(z) > 2^{-2/p}\nu \text{ for all } z \in \mathcal{O}. \quad (6.8)$$

With this goal in mind, consider $\bigcup_{z \in \mathcal{O}} B_{z_0/2}(z)$. All points of this set are at least $y_0/6$ away from the boundary of Ω and the diameter of this set is comparable to y_0 . Select the number $b > a$ so that

$$\mathcal{B} := \bigcup_{z \in \mathcal{O}} B_{z_0/2}(z) \subset \Gamma_b(0, x'). \quad (6.9)$$

A simple geometrical argument shows that b can be chosen independently of the location of points x', y' , and only depends on the size of a . Our goal is to estimate the difference $|w^{p/2}(z) - w^{p/2}(y)|$ for all $z \in \mathcal{O}$. To this end, fix some $z \in \mathcal{O}$. Abbreviating $B := B_{1/2}(0)$ then permits us to express

$$w^{p/2}(z) = \left(\int_B |u(z + z_0\xi)|^p d\xi \right)^{1/2}, \quad w^{p/2}(y) = \left(\int_B |u(y + y_0\xi)|^p d\xi \right)^{1/2}. \quad (6.10)$$

It follows that for $v = |u|^{p/2-1}u$ we have

$$\begin{aligned} w^{p/2}(z) &= \left(\int_B |v(y + y_0\xi) + [v(z + z_0\xi) - v(y + y_0\xi)]|^2 d\xi \right)^{1/2} \\ &\leq \left(\int_B |v(y + y_0\xi)|^2 d\xi \right)^{1/2} + \left(\int_B |v(z + z_0\xi) - v(y + y_0\xi)|^2 d\xi \right)^{1/2} \\ &= w^{p/2}(y) + \left(\int_B |v(z + z_0\xi) - v(y + y_0\xi)|^2 d\xi \right)^{1/2}. \end{aligned} \quad (6.11)$$

Since a similar estimate holds when the roles of y and z are interchanged, we eventually conclude that

$$|w^{p/2}(z) - w^{p/2}(y)|^2 \leq \int_B |v(z + z_0\xi) - v(y + y_0\xi)|^2 d\xi. \quad (6.12)$$

The fundamental theorem of calculus shows that for any two points $z_1, z_2 \in \mathcal{B}$ we have

$$\begin{aligned} |v(z_1) - v(z_2)|^2 &\leq \left| \int_0^1 (\nabla v)(z_1 + (z_2 - z_1)\tau) \cdot (z_1 - z_2) d\tau \right|^2 \\ &\leq |z_1 - z_2|^2 \int_0^1 |(\nabla v)(z_1 + (z_2 - z_1)\tau)|^2 d\tau \\ &= y_0^{n-2} |z_1 - z_2|^2 \int_0^1 |(\nabla v)(z_1 + (z_2 - z_1)\tau)|^2 y_0^{2-n} d\tau \\ &\leq C y_0^{n-1} \int_{[z_1, z_2]} (|u|^{p-2} |\nabla u|^2)(q) q_0^{2-n} ds(q), \end{aligned} \quad (6.13)$$

where the last integral is understood as a line integral over the segment joining z_1 and z_2 . We have also used Lemma 2.5 implying that $|\nabla v|^2 \approx |u|^{p-2} |\nabla u|^2$. We have also use the fact that $|z_1 - z_2| \leq C y_0$ for all $z_1, z_2 \in \mathcal{B}$. We apply this formula to generic pairs of points of the form $z + z_0\xi$, $y + y_0\xi$ for $z \in \mathcal{O}$ and $\xi \in B$ (which, by design, are in \mathcal{B}) and then integrate in ξ . Notice that, for various points ξ , the lines joining $z + z_0\xi$ with $y + y_0\xi$ are almost parallel; in fact they are genuinely parallel when $z_0 = y_0$. When integrating in ξ over B a typical point q in the very last expression in (6.13) considered with $z_1 := z + z_0\xi$ and $z_2 := y + y_0\xi$ will belong to certain line segments joining these points with ξ belonging to a certain subset of B of 1-dimensional Hausdorff measure, having size $O(1)$ relative to this measure. Hence,

$$\frac{1}{|B|} \int_B |v(z + z_0\xi) - v(y + y_0\xi)|^2 d\xi \leq C \int_{\mathcal{H}} (|u|^{p-2} |\nabla u|^2)(q) q_0^{2-n} dq, \quad (6.14)$$

where \mathcal{H} denotes the convex hull of the set $B_{z_0/2}(z) \cup B_{y_0/2}(y) \subset \Gamma_b(0, x')$, which is a set of diameter comparable to y_0 . The factor y_0^{n-1} in (6.13) disappears after integrating in ξ due to the natural change of variables which takes $ds(q)d\xi$ into dq in (6.14), the natural Lebesgue measure on \mathcal{H} . Because \mathcal{H} is contained in $\Gamma_b(0, x')$ the right-hand side of (6.14) may be further estimated by $S_{p,b}^p(u)(x') \leq \gamma^p \nu^p$. Hence, by combining (6.12)-(6.14) we obtain

$$|w^{p/2}(z) - w^{p/2}(y)|^2 \leq C(a, n, N)(\gamma \nu)^p \leq \frac{\nu^p}{4}, \quad (6.15)$$

if γ is chosen so that $C(a, n, N)\gamma^p < 1/4$. It follows that for any $z \in \mathcal{O}$ we have

$$w^{p/2}(z) \geq w^{p/2}(y) - |w^{p/2}(y) - w^{p/2}(z)| \geq \nu^{p/2} - \frac{\nu^{p/2}}{2} = \frac{\nu^{p/2}}{2}. \quad (6.16)$$

Hence $w(z) > 2^{-2/p}\nu$ and the claim in (6.8) follows, finishing the proof of the lemma. \square

Given a Lipschitz function $h : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, denote by M_h the Hardy-Littlewood maximal function considered on the graph of h . That is, given any locally integrable function f on the Lipschitz surface $\Lambda_h = \{(h(z'), z') : z' \in \mathbb{R}^{n-1}\}$, define $(M_h f)(x) := \sup_{r>0} \int_{\Lambda_h \cap B_r(x)} |f| d\sigma$ for each $x \in \Lambda_h$.

Corollary 6.3. *Let u be an energy solution of (4.1) in $\Omega = \mathbb{R}_+^n$ and fix $a > 0$. Associated with these, let b, γ be as in Lemma 6.2. Then there exists a finite constant $C = C(n, p) > 0$ with the property that for any $\nu > 0$ and any point $x' \in E_{\nu, a}$ such that $S_{p, b}(u)(x') \leq \gamma\nu$ one has*

$$(M_{h_{\nu, a}} w)(h_{\nu, a}(x'), x') \geq C\nu. \quad (6.17)$$

Lemma 6.4. *Consider the equation (4.1) with coefficients satisfying assumptions of Theorem 1.2. Then there exists $a > 1$ with the following significance. Suppose u is a weak solution of $\mathcal{L}u = 0$ in $\Omega = \mathbb{R}_+^n$. Select $\theta \in [1/6, 6]$ and, having picked $\nu > 0$ arbitrary, let $h_{\nu, a}(w)$ be as in (6.2). Also, consider the domain $\mathcal{O} = \{(x_0, x') \in \Omega : x_0 > \theta h_{\nu, a}(x')\}$ with boundary $\partial\mathcal{O} = \{(x_0, x') \in \Omega : x_0 = \theta h_{\nu, a}(x')\}$. In this context, for any surface ball $\Delta_r = B_r(Q) \cap \partial\Omega$, with $Q \in \partial\Omega$ and $r > 0$ chosen such that $h_{\nu, a}(w) \leq 2r$ pointwise on Δ_{2r} , one has*

$$\begin{aligned} \int_{\Delta_r} |u(\theta h_{\nu, a}(w)(\cdot), \cdot)|^p dx' &\leq C(1 + \|\mu\|_C^{1/2}) \|S_{p, b}(u)\|_{L^p(\Delta_{2r})}^{p/2} \|\tilde{N}_{p, a}(u)\|_{L^p(\Delta_{2r})}^{p/2} \\ &\quad + C \|S_{p, b}(u)\|_{L^p(\Delta_{2r})}^p + \frac{c}{r} \iint_{\mathcal{K}} |u|^p dX. \end{aligned} \quad (6.18)$$

Here $C = C(\lambda, \Lambda, n, N) \in (0, \infty)$ and \mathcal{K} is a region inside \mathcal{O} of diameter, distance to the boundary $\partial\mathcal{O}$, and distance to Q , are all comparable to r . Also, the parameter $b > a$ is as in Lemma 6.2, and the cones used to define the square and nontangential maximal functions in this lemma have vertices on $\partial\Omega$.

Moreover, the term $\iint_{\mathcal{K}} |u|^p dX$ appearing in (6.18) may be replaced by the quantity

$$Cr^{n-1} |\tilde{u}(A_r)|^2 + C \int_{\Delta_{2r}} S_{p, b}^p(u) d\sigma, \quad (6.19)$$

where A_r is any point inside \mathcal{K} (usually called a corkscrew point of Δ_r) and

$$\tilde{u}(X) := \int_{B_{\delta(X)/2}(X)} |u(Z)|^{p/2-1} u(Z) dZ. \quad (6.20)$$

Proof. Fix $\theta \in [1/6, 6]$. Consider the pullback transformation $\rho : \mathbb{R}_+^n \rightarrow \mathcal{O}$ defined as in section 3.3 relative to the Lipschitz function $h_{\nu, a}(w)$. Let v be given by $v := u \circ \rho$ in \mathbb{R}_+^n . Thanks to the assumptions made on the PDE (4.1), the function $v : \mathbb{R}_+^n \rightarrow \mathbb{R}$ will satisfy a PDE similar to that of u . Specifically, we have

$$\partial_i (\bar{A}_{ij}(x) \partial_j v) + \bar{B}_i(x) \partial_i v = 0, \quad (6.21)$$

where \bar{A} is elliptic and the coefficients \bar{A} and \bar{B} are such that

$$d\bar{\mu}(x) = \left[\left(\sup_{B_{\delta(x)/2}(x)} |\nabla \bar{A}(x)| \right)^2 + \left(\sup_{B_{\delta(x)/2}(x)} |\bar{B}(x)| \right)^2 \right] \delta(x) dx \quad (6.22)$$

is a Carleson measure in \mathbb{R}_+^n . Moreover, the Carleson norm $\|\bar{\mu}\|_C$ only depends on the Carleson norm of the original coefficients and a (the aperture of nontangential cones).

The parameter $a > 1$ may be chosen large enough so that the Lipschitz constant of the function $\theta h_{\nu,a}$ is small (at most $6/a$). As we have observed for the original equation we may arrange (by change of variables) that the coefficient $\bar{A}_{00} = 1$.

Having fixed a scale $r > 0$, we localize to a ball $B_r(y')$ in \mathbb{R}^{n-1} . Let ζ be a smooth cutoff function of the form $\zeta(x_0, x') = \zeta_0(x_0)\zeta_1(x')$ where

$$\zeta_0 = \begin{cases} 1 & \text{in } [0, r], \\ 0 & \text{in } [2r, \infty), \end{cases} \quad \zeta_1 = \begin{cases} 1 & \text{in } B_r(y'), \\ 0 & \text{in } \mathbb{R}^n \setminus B_{2r}(y') \end{cases} \quad (6.23)$$

and

$$r|\partial_0 \zeta_0| + r|\nabla_{x'} \zeta_1| \leq c \quad (6.24)$$

for some constant $c \in (0, \infty)$ independent of r . Our goal is to control the L^p norm of $u(\theta h_{\nu,a}(w)(\cdot), \cdot)$. Since after the pullback under the mapping ρ the latter is comparable with the L^p norm of $v(0, \cdot)$, we proceed to estimate

$$\begin{aligned} & \int_{B_{2r}(y')} |v|^p(0, x') \zeta(0, x') dx' \\ &= - \iint_{[0, 2r] \times B_{2r}(y')} \partial_0 [|v|^p \zeta](x_0, x') dx_0 dx' \\ &= -p \iint_{[0, 2r] \times B_{2r}(y')} |v|^{p-2} \Re e \langle v, \partial_0 v \rangle \zeta dx_0 dx' \\ &\quad - \iint_{[0, 2r] \times B_{2r}(y')} |v|^p(x_0, x') \partial_0 \zeta dx_0 dx' \\ &=: \mathcal{A} + V. \end{aligned} \quad (6.25)$$

We further expand the term \mathcal{A} as a sum of three terms obtained via integration by parts with respect to x_0 as follows:

$$\begin{aligned}
\mathcal{A} &= -p \iint_{[0,2r] \times B_{2r}(y')} |v|^{p-2} \mathcal{R}e \langle v, \partial_0 v \rangle \zeta (\partial_0 x_0) dx_0 dx' \\
&= p(p-2) \iint_{[0,2r] \times B_{2r}(y')} |v|^{p-2} |\mathcal{R}e \langle v, \partial_0 v \rangle|^2 x_0 \zeta dx_0 dx' \\
&\quad + p \iint_{[0,2r] \times B_{2r}(y')} |v|^{p-2} |\partial_0 v|^2 x_0 \zeta dx_0 dx' \\
&\quad + p \iint_{[0,2r] \times B_{2r}(y')} |v|^{p-2} \mathcal{R}e \langle v, \partial_{00}^2 v \rangle x_0 \zeta dx_0 dx' \\
&\quad + p \iint_{[0,2r] \times B_{2r}(y')} |v|^{p-2} \mathcal{R}e \langle v, \partial_0 v \rangle x_0 \partial_0 \zeta dx_0 dx' \\
&=: I + II + III + IV.
\end{aligned} \tag{6.26}$$

We start by analyzing the term III . In view of the fact that $\bar{A}_{00} = 1$, the PDE recorded in (6.21) allows us to write

$$\partial_{00}^2 v = - \sum_{(i,j) \neq (0,0)} \partial_i (\bar{A}_{ij} \partial_j v) - B_i \partial_i v. \tag{6.27}$$

In turn, this permits us to express

$$\begin{aligned}
III &= -p \mathcal{R}e \sum_{(i,j) \neq (0,0)} \iint_{[0,2r] \times B_{2r}} (\partial_i \bar{A}_{ij}) |v|^{p-2} \bar{v} \partial_j v x_0 \zeta dx_0 dx' \\
&\quad - p \mathcal{R}e \iint_{[0,2r] \times B_{2r}} B_i |v|^{p-2} \bar{v} \partial_i v x_0 \zeta dx_0 dx' \\
&\quad - p \mathcal{R}e \sum_{(i,j) \neq (0,0)} \iint_{[0,2r] \times B_{2r}} \bar{A}_{ij} |v|^{p-2} \bar{v} \partial_{ij}^2 v x_0 \zeta dx_0 dx' \\
&=: III_1 + III_2 + III_3.
\end{aligned} \tag{6.28}$$

The last term above requires some further work. Let us temporarily fix i, j and denote by III_3^{ij} the corresponding term in III_3 . Since in the present context we have $(i, j) \neq (0, 0)$, at least one of the two indices involved is not zero, say $i > 0$. Integrating by parts with respect to the variable x_i then yields (in what follows we do not sum over indices i and j)

$$\begin{aligned}
III_3^{ij} &= p \mathcal{R}e \iint_{[0,2r] \times B_{2r}} (\partial_i \bar{A}_{ij}) |v|^{p-2} \bar{v} \partial_j v x_0 \zeta dx_0 dx' \\
&\quad + p \mathcal{R}e \iint_{[0,2r] \times B_{2r}} \bar{A}_{ij} \partial_i (|v|^{p-2} \bar{v}) \partial_j v x_0 \zeta dx_0 dx' \\
&\quad + p \mathcal{R}e \iint_{[0,2r] \times B_{2r}} \bar{A}_{ij} |v|^{p-2} \bar{v} \partial_j v x_0 \partial_i \zeta dx_0 dx' \\
&= J_1^{ij} + J_2^{ij} + J_3^{ij}.
\end{aligned} \tag{6.29}$$

The treatment of III_3^{ij} in the case when $i = 0$ proceeds along the same lines, except that we now integrate in the variable x_j . Since the resulting terms are of a similar nature as above, we omit writing them explicitly.

We now group together terms that are of the same type. Firstly, we have

$$I + II + J_2 \leq C(\lambda, \Lambda, n, N) \|S_{p,b}(u)\|_{L^p(B_{2r})}^p. \quad (6.30)$$

Here, the estimate would be true even with truncated p -adapted square function $\|S_{p,b}^{2r}(v)\|_{L^p(B_{2r})}^p$ which is at every point dominated by $\|S_{p,b}(u)\|_{L^p(B_{2r})}^p$. Secondly, the Carleson condition (6.22) and the Cauchy-Schwarz inequality imply

$$III_1 + III_2 + J_1 \leq C(n, N) \|\mu\|_{\mathcal{C}}^{1/2} \|S_{p,b}(u)\|_{L^p(B_{2r})}^{p/2} \|\tilde{N}_{p,a}(u)\|_{L^p(B_{2r})}^{p/2}. \quad (6.31)$$

Next, corresponding to the case when the derivative falls on the cutoff function ζ we have

$$\begin{aligned} J_3 + IV &\leq C(\lambda, \Lambda, n, N) \iint_{[0,2r] \times B_{2r}} |\nabla v| |v|^{p-1} \frac{x_0}{r} dx_0 dx' \\ &\leq C(\lambda, \Lambda, n, N) \left(\iint_{[0,2r] \times B_{2r}} |v|^p \frac{x_0}{r^2} dx_0 dx' \right)^{1/2} \|S_{p,b}^{2r}(v)\|_{L^p(B_{2r})}^{p/2} \\ &\leq C(\lambda, \Lambda, n, N) \|S_{p,b}(u)\|_{L^p(B_{2r})}^{p/2} \|\tilde{N}_{p,a}(u)\|_{L^p(B_{2r})}^{p/2}. \end{aligned} \quad (6.32)$$

Finally, the interior term V , which arises from the fact that $\partial_0 \zeta$ vanishes on the set $(0, r) \cup (2r, \infty)$ may be estimated as follows:

$$V \leq \frac{c}{r} \iint_{[r,2r] \times B_{2r}} |v|^p dx_0 dx'. \quad (6.33)$$

Summing up all terms, the above analysis ultimately yields

$$\begin{aligned} &\int_{B_r(y')} |v(0, x')|^p dx' \\ &\leq C(\lambda, \Lambda, n, N) (1 + \|\mu\|_{\mathcal{C}}^{1/2}) \|S_{p,b}(u)\|_{L^p(B_{2r})}^{p/2} \|\tilde{N}_a(u)\|_{L^p(B_{2r})}^{p/2} \\ &\quad + C(\lambda, \Lambda, n, N) \|S_{p,b}(u)\|_{L^p(B_{2r})}^p + \frac{c}{r} \iint_{[r,2r] \times B_{2r}} |v|^p dx_0 dx'. \end{aligned} \quad (6.34)$$

With this in hand, the estimate in (6.18) follows (by passing from v back to u via the map ρ).

Finally, the last claim in the statement of the lemma can be seen as follows. If $\mathcal{K} = B_{\delta(X)/2}(X)$ and $A_r = X$ then the claim in question is easy to prove as it is just the Poincaré inequality for the function $\tilde{v} = |u|^{p/2-1}u$. We have

$$\int_B |u|^p dX = \int_B |\tilde{v}|^2 dX \leq Cr^n \left(\oint_B \tilde{v} dX \right)^2 + Cr^2 \int_B |\nabla \tilde{v}|^2 dX.$$

The first term is just $Cr^n |\tilde{u}(A_r)|^2$, while the second can be bounded by $Cr^n (S_{p,b}(u)(Q))^p$ for all $Q \in \Delta_{2r}$. From this the estimate by $r \int_{\Delta_{2r}} S_{p,b}^p(u) d\sigma$ follows.

For more general sets \mathcal{K} , there is a finite covering of \mathcal{K} by balls of the form $B_i = B_{\delta(X_i)/2}(X_i)$. Then

$$r^{-1} \iint_{\mathcal{K}} |u|^p dX \leq r^{-1} \sum_i \int_{B_i} |u|^p dZ \leq C \sum_i r^{n-1} |\tilde{u}(X_i)|^p + \int_{\Delta_{2r}} S_{p,b}^p(u) d\sigma, \quad (6.35)$$

by the previous argument for the balls. For each i we have (abbreviating $r_i := \delta(X_i)$, $\bar{r} := \delta(A_r)$, and $B := B_{1/2}(0)$):

$$\begin{aligned} |\tilde{u}(X_i)|^p &= |\tilde{v}(X_i)|^2 \leq 2|\tilde{v}(A_r)|^2 + 2|\tilde{v}(X_i) - \tilde{v}(A_r)|^2 \\ &= 2|\tilde{u}(A_r)|^p + 2 \left(\int_B |\tilde{v}(X_i + r_i \xi) - \tilde{v}(A_r + \bar{r} \xi)| d\xi \right)^2 \\ &\leq 2|\tilde{u}(A_r)|^p + 2 \int_B |\tilde{v}(X_i + r_i \xi) - \tilde{v}(A_r + \bar{r} \xi)|^2 d\xi. \end{aligned} \quad (6.36)$$

Note that the last term above is of the same type as the right-hand side of (6.12). As we did there, the term in question may once again be estimated as in (6.14). Hence, ultimately, this is $\leq C(S_{p,b}(u)(Q))^p$ for all $Q \in \Delta_{2r}$. The desired conclusion now readily follows from this. \square

We now make use of Lemma 6.4, involving the stopping time Lipschitz functions $\theta h_{\nu,a}(w)$, in order to obtain the good- λ inequality stated in the next lemma. As a preamble, we agree to let $Mf(x') := \sup_{r>0} \int_{|x'-z'|<r} |f(z')| dz'$, for $x' \in \mathbb{R}^{n-1}$, denote the standard Hardy-Littlewood maximal function on $\partial \mathbb{R}_+^n = \mathbb{R}^{n-1}$.

Lemma 6.5. *Consider the PDE (4.1) with coefficients satisfying assumptions of Theorem 1.2 in \mathbb{R}_+^n . Then for each $\gamma \in (0, 1)$ there exists a constant $C(\gamma) > 0$ such that $C(\gamma) \rightarrow 0$ as $\gamma \rightarrow 0$ and with the property that for each $\nu > 0$ and each energy solution u of (4.1) there holds*

$$\begin{aligned} &\left| \left\{ x' \in \mathbb{R}^{n-1} : \tilde{N}_a(u) > \nu, (M(S_{p,b}^p(u)))^{1/p} \leq \gamma\nu, (M(S_{p,b}^p(u))M(\tilde{N}_{p,a}^p(u)))^{1/2p} \leq \gamma\nu \right\} \right| \\ &\leq C(\gamma) \left| \left\{ x' \in \mathbb{R}^{n-1} : \tilde{N}_{p,a}(u)(x') > \nu/32 \right\} \right|. \end{aligned} \quad (6.37)$$

Proof. We first observe that $\{x' \in \mathbb{R}^{n-1} : \tilde{N}_{p,a}(u)(x') > \nu/32\}$ is an open subset of \mathbb{R}^{n-1} . When this set is empty, or is all of \mathbb{R}^{n-1} , estimate (6.37) is trivial, so we focus on the case when the set in question is both nonempty and proper. Granted this, we may consider a Whitney decomposition $(\Delta_i)_{i \in I}$ of it, consisting of open cubes in \mathbb{R}^{n-1} . Let F_ν^i be the set appearing on the left-hand side of (6.37) intersected with Δ_i . We may streamline the index set I by retaining only those i 's for which $F_\nu^i \neq \emptyset$. Let B_i be a ball of radius r_i in \mathbb{R}^n such that $\Delta_i \subset B_i \cap \{x_0 = 0\}$ and there exists a point $p' \in 2B_i \cap \partial \mathbb{R}_+^n$ with $\tilde{N}_{p,a}(u)(p') = N_a(w)(p') \leq \nu/32$. The existence of such point p' is guaranteed by the very nature of the Whitney decomposition. Indeed, there exists a point near Δ_i not contained in the set $\{x' \in \mathbb{R}^{n-1} : \tilde{N}_{p,a}(u)(x') > \nu/32\}$.

This clearly implies that $w(z) \leq \nu/32$ for all $z \in \Gamma_a(p')$. In particular, for all $x' \in \Delta_i$ we have $w(z) \leq \nu/32$ for all $z \in \Gamma_a(x') \cap \Gamma_a(p')$, so we focus on estimating the size of $w(z)$ for $z \in \Gamma_a(x') \setminus \Gamma_a(p')$ with $z_0 \geq 2r$. Since we also assume that for at least one $x' \in \Delta_i$ we have $M(S_{p,b}^p(u))(x') \leq (\gamma\nu)^p$, by the same type of estimates

established in the proof of Lemma 6.2 (and (6.15) in particular), we may conclude that we have that for any $z \in \Gamma_a(x')$ with $z_0 \geq 2r$ there is a point $\tilde{z} \in \Gamma_a(p')$ with

$$|z - \tilde{z}| \leq Cr_i \quad \text{and} \quad |w^{p/2}(z) - w^{p/2}(\tilde{z})| \leq c(\gamma)\nu^{p/2}. \quad (6.38)$$

It follows that for γ chosen sufficiently small, for all such z , we have the estimate $w(z) \leq \nu/16$. Hence for all $x' \in \Delta_i$ we have

$$\nu < \tilde{N}_{p,a}(u)(x') = N_a(w)(x') = N_a^{2r}(w)(x'), \quad (6.39)$$

where N_a^{2r} is the truncated nontangential maximal function at height $2r$. In particular this also implies

$$h_{\nu,a}(w) \leq 2r_i \quad \text{pointwise on } \Delta_i. \quad (6.40)$$

Let us also note that we can find a point q (specifically, a corkscrew point for $12\Delta_i$) with distance to Δ_i and the boundary equal to $12r_i$ such that $w(q) \leq \nu/16$. Recall the definition of \tilde{u} by (6.20). Clearly,

$$|\tilde{u}(q)| \leq w(q) \leq \nu/16. \quad (6.41)$$

Lemma 6.6. *There exists a complex number $u_0 \in \mathbb{C}$ with $|u_0| \leq \nu/16$ such that for $v := u - u_0$ we have $\mathcal{L}v = 0$, hence v still solves the equation (4.1) and $\tilde{v}(q) = 0$, where*

$$\tilde{v}(x) := \oint_{B_{\delta(x)/2}(x)} |v(z)|^{p/2-1} v(z) dz.$$

We postpone the proof of this claim for later. Assuming the claim, let us denote by \tilde{w} the L^p averages of $|v|$. For all $x' \in F_\nu^i$ we have

$$N_a^{2r}(\tilde{w})(x') \geq N_a^{2r}(w)(x') - |u_0| \geq \nu - \nu/16 > \nu/2. \quad (6.42)$$

With $h := h_{\nu,a}(\tilde{w})$ and for M_h defined on the graph of h in Corollary 6.3 we see that Corollary 6.3 applied to v implies¹

$$M_h(\tilde{w}\chi_{4B_i})(h(x'), x') \geq C(n)\nu. \quad (6.43)$$

Here we are allowed to apply the cutoff function χ_{4B_i} since values of \tilde{w} are small above the height $2r$, and hence this put a limit on the distance and the diameter of the boundary ball R constructed in Corollary 6.3 from the point x' (both are bounded by $\lesssim r_i$). Thus by the maximal function theorem

$$\begin{aligned} |F_\nu^i| &\leq \frac{C}{\nu^p} \int_{4\Delta_i} (M_h(\tilde{w}\chi_{4B_i}))^p(h(x'), x') dx' \\ &\leq \frac{C}{\nu^p} \int_{4\Delta_i} \tilde{w}^p(h(x'), x') dx'. \end{aligned} \quad (6.44)$$

At this stage, we will need the following lemma.

Lemma 6.7. *For any surface ball Δ if $a > 1$ we have for $h = h_{\nu,a}(\tilde{w})$*

$$\int_{\Delta} \tilde{w}^p(h(x'), x') dx' \leq C \int_{1/6}^6 \int_{3\Delta} |v(\theta h(x'), x')|^p dx' d\theta. \quad (6.45)$$

¹Technically $v \in W_{loc}^{1,2}(\Omega)$ is not an energy solution, but in the proof the smallness of the solution is only need above a certain distance from the boundary. In our case we obviously have $\tilde{w}(z) \leq w(z) + |u_0| \leq \nu/8$ for points z whose distance to the boundary exceeds $2r_i$ which suffices for our purposes.

Accepting the lemma, whose proof we postpone until later, we have (taking $a > 1$ as in Lemma 6.4)

$$|F_\nu^i| \leq \frac{C}{\nu^p} \int_{1/6}^6 \int_{12\Delta_i} |v(\theta h(x'), x')|^p dx' d\theta. \quad (6.46)$$

For each θ we apply the conclusion in Lemma 6.4 (in the version recorded in the very last part of its statement) to the solution v . This gives

$$\begin{aligned} & \int_{12\Delta_i} |v(\theta h(x'), x')|^p dx' \\ & \leq C(1 + \|\mu\|_C^{1/2}) \|S_{p,b}(u)\|_{L^p(24\Delta_i)}^{p/2} \|N_a(\tilde{w})\|_{L^p(24\Delta_i)}^{p/2} \\ & \quad + C \|S_{p,b}(u)\|_{L^p(24\Delta_i)}^p + Cr^{n-1} |\tilde{v}(q)|^2 \\ & \leq C(1 + \|\mu\|_C^{1/2}) \|S_{p,b}(u)\|_{L^p(24\Delta_i)}^{p/2} \|N_a(w) + |u_0|\|_{L^p(24\Delta_i)}^{p/2} \\ & \quad + C \|S_{p,b}(u)\|_{L^p(24\Delta_i)}^p. \end{aligned} \quad (6.47)$$

Observe that we have dropped the term $Cr^{n-1} |\tilde{v}(q)|^2$, since we have arranged previously that $\tilde{v}(q) = 0$. As $F_\nu^i \neq \emptyset$ and $|u_0| \leq \nu/16$, the term in the penultimate line of (6.47) may be bounded by a constant times

$$\begin{aligned} & |24\Delta_i| \left(\int_{24\Delta_i} S_{p,b}^p(u) dx' \right)^{p/2} \left[\left(\int_{24\Delta_i} N_{p,a}^p(u) dx' \right)^{p/2} + \left(\frac{\nu}{16} \right)^{p/2} \right] \\ & \leq C |24\Delta_i| \left[(M(S_{p,b}^p(u))(x') M(\tilde{N}_{p,a}^p(u))(x'))^{1/2} + \left(\frac{\nu}{16} \right)^{p/2} M(S_{p,b}^p(u))(x')^{1/2} \right] \\ & \leq C |24\Delta_i| (\gamma^p + (\gamma/16)^{p/2}) \nu^p = C(\gamma) |\Delta_i| \nu^p. \end{aligned} \quad (6.48)$$

Here $x' \in F_\nu^i$ is a point where we use the assumptions for the set on the left-hand side of (6.37). Also, we have used that $|24\Delta_i| \lesssim |\Delta_i|$ by the doubling property of the Lebesgue measure. The estimate for the very last term of (6.47) is analogous. By design, we have $C(\gamma) \rightarrow 0$ as $\gamma \rightarrow 0$. Using this back in (6.46) we obtain

$$|F_\nu^i| \leq C'(\gamma) |\Delta_i|. \quad (6.49)$$

Summing over all i we obtain (6.37), as desired. \square

At this stage, it remains to prove Lemmas 6.6 and 6.7. We start with the second one.

Proof. Write $\mathbb{R}^{n-1} = \bigcup_{i \in \mathbb{Z}} \Delta^i$ where, for each i ,

$$\Delta^i := \{x' \in \mathbb{R}^{n-1} : 2^{i-1} \leq h(x') < 2^i\}. \quad (6.50)$$

Consider $y = (y_0, y') \in B_{h(x')/2}(h(x'), x')$ for $x' \in \Delta^i \cap \Delta$. Then

$$|y_0 - h(x')| \leq h(x')/2 \quad \text{and} \quad |x' - y'| \leq h(x')/2. \quad (6.51)$$

The goal is to estimate $h(y')$. Since $h = h_{\nu,a}(w)$ is a Lipschitz function with Lipschitz constant $1/a < 1$ (cf. Lemma 6.1) we have

$$h(y') \geq h(x') - |h(x') - h(y')| \geq h(x') - |x' - y'| > \frac{h(x')}{2} \quad (6.52)$$

and

$$h(y') \leq h(x') + |h(x') - h(y')| \leq h(x') + |x' - y'| < 3h(x')/2. \quad (6.53)$$

It follows that if $\mathcal{O} := \bigcup_{x' \in \Delta^i \cap \Delta} B_{h(x')/2}(h(x'), x')$ then

$$(y_0, y') \in \mathcal{O} \implies \begin{cases} y' \in \widetilde{\Delta}^i := \Delta^{i-1} \cup \Delta^i \cup \Delta^{i+1}, \\ y' \in 3\Delta, \\ y_0 \in [2^{i-2}, 3 \cdot 2^{i-1}). \end{cases} \quad (6.54)$$

The fact that $y' \in 3\Delta$ follows from (6.40). Hence we have

$$\begin{aligned} & \int_{\Delta^i \cap \Delta} \tilde{w}^p(h(x'), x') dx' \\ &= \int_{x' \in \Delta^i \cap \Delta} \int_{B_{h(x')/2}(h(x'), x')} |v(z)|^p dz dx' \\ &\leq C 2^{-in} \int_{x' \in \Delta^i \cap \Delta} \int_{B_{h(x')/2}(h(x'), x')} |v(z)|^p dz dx' \\ &= C 2^{-in} \iint_{\mathcal{O}} |v(z)|^p |\{x' \in \Delta^i \cap \Delta : z \in B_{h(x')/2}(h(x'), x')\}| dz, \end{aligned} \quad (6.55)$$

where in the last step we have interchanged the order of integration. For a fixed $z \in \mathcal{O}$ we have

$$\begin{aligned} & |\{x' \in \Delta^i \cap \Delta : z \in B_{h(x')/2}(h(x'), x')\}| \\ &\leq |\{x' \in \mathbb{R}^{n-1} : z \in B_{h(x')/2}(h(x'), x')\}|. \end{aligned} \quad (6.56)$$

Since for such $z = (z_0, z')$ we have $z_0 \in [2^{i-2}, 3 \cdot 2^{i-1})$ and

$$\frac{h(x')}{2} < z_0 < \frac{3h(x')}{2} \implies h(x') \in (2z_0/3, 2z_0) \subset (2^{i-1}/3, 3 \cdot 2^i). \quad (6.57)$$

From this we then conclude

$$\{x' \in \mathbb{R}^{n-1} : z \in B_{h(x')/2}(h(x'), x')\} \subset \{x' \in \mathbb{R}^{n-1} : |x' - z'| < 2^{i+2}\} \quad (6.58)$$

hence, further,

$$|\{x' \in \Delta^i \cap \Delta : z \in B_{h(x')/2}(h(x'), x')\}| \leq C 2^{i(n-1)}. \quad (6.59)$$

Using this back in (6.55) then yields

$$\begin{aligned} & \int_{\Delta^i \cap \Delta} \tilde{w}^p(h(x'), x') dx' \leq C 2^{-i} \iint_{\mathcal{O}} |v(z)|^p dz \\ &\leq C \int_{z' \in \mathcal{P}(\mathcal{O})} \int_{z_0 \in (2^{i-2}, 3 \cdot 2^{i-1})} |v(z)|^p dz_0 dz', \end{aligned} \quad (6.60)$$

where (with $\widetilde{\Delta}^i$ as in (6.54))

$$\mathcal{P}(\mathcal{O}) := \{z' \in \mathbb{R}^{n-1} : \exists z_0 \text{ such that } (z_0, z') \in \mathcal{O}\} \subset \widetilde{\Delta}^i \cap 3\Delta. \quad (6.61)$$

Clearly since for $z' \in \mathcal{P}(\mathcal{O})$ we have $h(z') \in [2^{i-2}, 3 \cdot 2^{i-1})$ and, therefore,

$$(2^{i-2}, 3 \cdot 2^{i-1}) \subset (h(z')/6, 6h(z')). \quad (6.62)$$

Hence (6.60) may be also written as

$$\int_{\Delta^i \cap \Delta} \tilde{w}^p(h(x'), x') dx' \leq C \int_{\widetilde{\Delta^i} \cap 3\Delta} \int_{1/6}^6 |v(\theta h(z'), z')|^p d\theta dz'. \quad (6.63)$$

By interchanging the order of integration and then summing over all $i \in \mathbb{Z}$ we arrive at

$$\begin{aligned} & \int_{\Delta} \tilde{w}^p(h(x'), x') dx' \\ & \leq C \int_{1/6}^6 \sum_i \int_{\widetilde{\Delta^i} \cap 3\Delta} |v(\theta h(z'), z')|^p dz' d\theta \\ & = C \int_{1/6}^6 \sum_i \left(\int_{\Delta^{i-1} \cap 3\Delta} + \int_{\Delta^i \cap 3\Delta} + \int_{\Delta^{i+1} \cap 3\Delta} \right) |v(\theta h(z'), z')|^p dz' d\theta \\ & = 3C \int_{1/6}^6 \sum_i \int_{\Delta^i \cap 3\Delta} |v(\theta h(z'), z')|^p dz' d\theta \\ & = 3C \int_{1/6}^6 \int_{3\Delta} |v(\theta h(z'), z')|^p dz' d\theta, \end{aligned} \quad (6.64)$$

as wanted. This finishes the proof of Lemma 6.7. \square

We now sketch the proof of Lemma 6.6 without too much details, as it is essentially an exercise in degree theory.

Proof. Consider a continuous function $F : \mathbb{C} \rightarrow \mathbb{C}$ defined as follows:

$$F(c) = \oint_{B_{\delta(q)/2}(q)} |u(z) - c|^{p/2-1} (u(z) - c) dz.$$

We claim that for some $c \in \mathbb{C}$ we have $F(c) = 0$. If not then

$$H(c) = \frac{F(c)}{|F(c)|} \quad \text{is a continuous function } \mathbb{C} \rightarrow \{x \in \mathbb{C} : |x| = 1\}.$$

Clearly, all the maps $H_r = H|_{\{x \in \mathbb{C} : |x|=r\}} \rightarrow \{x \in \mathbb{C} : |x| = 1\}$ for $r > 0$ are homotopic as H is continuous. Since for all sufficiently large r (of the order of $w(q) = \oint_{B_{\delta(q)/2}(q)} |u(z)|^{p/2} dz$) the degree of H_r is 1 and hence H_r is onto. As all H_r , $r > 0$ are homotopic, they all must be onto. It follows that any neighbourhood of 0 is mapped by H onto $\{x \in \mathbb{C} : |x| = 1\}$ and hence H cannot be continuous at $0 \in \mathbb{C}$. This is a contradiction and therefore $F(c) = 0$ somewhere. It follows from the argument above that for such c we must have $|c| \lesssim w(q)$. \square

We can now state the bound of the nontangential maximal function by the square function.

Proposition 6.8. *Let u be an arbitrary energy solution of (4.1) in $\Omega = \mathbb{R}_+^n$. Assume that A is elliptic and the measure μ defined as in (1.12) is Carleson with norm $\|\mu\|_C < \infty$. Then for any $p, q \in (1, \infty)$ and $a > 1$ there exists a finite constant $C = C(n, N, \lambda, \Lambda, p, q, a, \|\mu\|_C) > 0$ such that*

$$\|\tilde{N}_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})} \leq C \|S_{p,a}(u)\|_{L^q(\mathbb{R}^{n-1})}. \quad (6.65)$$

Moreover, a local version of this estimate holds as well. Specifically, for any weak solution u of (4.1) and any boundary ball Δ_r one has

$$\|\tilde{N}_{p,a}^r(u)\|_{L^q(\Delta_r)}^q \leq C\|S_{p,a}^{2r}(u)\|_{L^q(2\Delta_r)}^q + Cr^{n-1}|\tilde{u}(A_r)|^{2q/p}, \quad (6.66)$$

where A_r is a corkscrew point relative to Δ_r and $\tilde{u} := \oint_{B_{\delta(A_r)/2}(A_r)} |u(z)|^{p/2-1} u(z) dz$.

Proof. We will be brief as the hard part was the good- λ inequality (Lemma 6.5). For $q > p$ (6.65) follows immediately by a standard argument from the good- λ inequality. For more details see [20]. Furthermore as in [20, Theorem 3.18] the global result for $q > p$ implies the local estimate (6.66) for some $q > q_0$. Finally the local estimate for all $q > 1$ then follows by a standard argument from the local one for some $q' > q_0$. See [16] for full details. \square

7. APPENDIX: THE BOUNDARY VALUES OF SOLUTIONS WITH $\|\tilde{N}_{2,a}(u)\|_{L^p} < \infty$.

The results in this section are of a general nature, and have applications to issues of nontangential convergence of solutions in the boundary value problems considered in this paper.

We start by considering an energy solution $u \in W^{1,2}(\Omega; \mathbb{C})$ of the Dirichlet boundary value problem (4.1). Denote by $\tilde{u} : \Omega \rightarrow \mathbb{C}$ the averages

$$\tilde{u}(x) = \oint_{B_{\delta(x)/2}(x)} u(y) dy, \quad \forall x \in \Omega.$$

Clearly, \tilde{u} is a continuous function on Ω . We shall establish the following lemma.

Lemma 7.1. *For $u \in W^{1,2}(\Omega; \mathbb{C})$ let $f = \text{Tr } u$ be its trace on $\partial\Omega$ (which belongs to the space $B_{1/2}^{2,2}(\partial\Omega; \mathbb{C})$). Then*

$$f(Q) = \lim_{x \rightarrow Q, x \in \Gamma(Q)} \tilde{u}(x), \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } Q \in \partial\Omega. \quad (7.1)$$

Proof. It suffices to work on $\Omega = \mathbb{R}_+^n$ since the pull-back transformation (3.27) defines a bijection between $W^{1,2}(\Omega; \mathbb{C})$ and $W^{1,2}(\mathbb{R}_+^n; \mathbb{C})$ and maps an interior ball $B_{\delta(x)/2}(x)$ in Ω into an open set on \mathbb{R}_+^n that contains and is contained in balls of radius comparable to $\delta(x)/2$. Hence the result proven on \mathbb{R}_+^n transfers to Ω .

Hence from now on let $\Omega = \mathbb{R}_+^n$. Writing $x \in \mathbb{R}_+^n$ as (x_0, x') consider the functions

$$\tilde{u}_k(x') = \tilde{u}(2^{-k}, x'), \quad \forall x' \in \mathbb{R}^{n-1}.$$

Then for any $x, y \in \mathbb{R}_+^n$ with $|x - y| \leq r$ and $\delta(x), \delta(y) \approx r$ we have

$$|\tilde{u}(x) - \tilde{u}(y)|^2 \lesssim \int_{\mathcal{H}} |\nabla u|^2 r^{2-n} dy \quad (7.2)$$

where \mathcal{H} is the convex hull of the set $B_{\delta(x)/2}(x) \cup B_{\delta(y)/2}(y)$. It follows that

$$\int_{\mathbb{R}^{n-1}} |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')|^2 dx' \lesssim \int_{(2^{-(k-1)}, 2^{-(k+2)}) \times \mathbb{R}^{n-1}} |\nabla u|^2 (2^{-k}) dy \leq 2^{-k} \|\nabla u\|_{L^2(\mathbb{R}_+^n)}^2.$$

From this we have that $(\tilde{u}_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^{n-1})$ and hence convergent. As f is the trace of u it follows that $\tilde{u}_k \rightarrow f$ in L^2 .

Next we show that $\tilde{u}_k \rightarrow f$ pointwise almost everywhere. For any $\lambda > 0$ consider the set

$$E_\lambda = \left\{ x' \in \mathbb{R}^{n-1} : \forall k \in \mathbb{N} \text{ we have } |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')|^2 \leq \frac{\lambda}{2^{k/2}} \right\}.$$

We estimate the size of the complement of E_λ . Clearly,

$$\begin{aligned} |E_\lambda^c| &\leq \sum_{k=1}^{\infty} \left| \left\{ x' \in \mathbb{R}^{n-1} : |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')|^2 > \frac{\lambda}{2^{k/2}} \right\} \right| \\ &\leq \sum_{k=1}^{\infty} \frac{2^{k/2}}{\lambda} \int_{\mathbb{R}^{n-1}} |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')|^2 dx' \leq \sum_{k=1}^{\infty} \frac{2^{k/2}}{\lambda} 2^{-k} \|\nabla u\|_{L^2(\mathbb{R}_+^n)}^2 \leq \frac{C}{\lambda}. \end{aligned}$$

It follows that $\cap_{\lambda>0} E_\lambda^c$ is a set of measure zero. Hence the set

$$\mathcal{S} = \left(\bigcup_{\lambda>0} E_\lambda \right) \cap \{x' \in \mathbb{R}^{n-1} : S_{10a}^1(u)(x') < \infty\}$$

has full measure. Here $S_{10a}^1(u)$ is the truncated square function at the height 1. Clearly, $\{x' \in \mathbb{R}^{n-1} : S_{10a}^1(u)(x') < \infty\}$ is a set of full measure due to our assumption that $u \in W^{1,2}(\mathbb{R}_+^n)$. Indeed,

$$\int_{\mathbb{R}^{n-1}} S_{10a}^1(u)(x') dx' \approx \int_{\mathbb{R}^{n-1} \times (0,1)} |\nabla u|^2 x_0 dx_0 dx' \leq \|\nabla u\|_{L^2(\mathbb{R}_+^n)}^2 < \infty,$$

and hence $S_{10a}^1(u)(x') < \infty$ a.e. as claimed and therefore \mathcal{S} is a set of full measure.

Consider any $y \in \Gamma_a^1(x')$ for $x' \in \mathcal{S}$. Find an integer $k \in \mathbb{N}$ such that $\delta(y) \approx 2^{-k}$ and hence also $|y - (2^{-k}, x')| \approx 2^{-k}$. We estimate the difference $\tilde{u}(y) - \tilde{u}_k(x')$. As before we have (c.f. (7.2))

$$|\tilde{u}(y) - \tilde{u}_k(x')|^2 \lesssim \int_{\mathcal{H}} |\nabla u(y)|^2 \delta(y)^{2-n} dy \lesssim \int_{\mathcal{O}_{k-2} \cup \mathcal{O}_{k-1} \cup \mathcal{O}_k \cup \mathcal{O}_{k+1}} |\nabla u(y)|^2 \delta(y)^{2-n} dy$$

where \mathcal{H} is the convex hull of the set $B_{\delta((2^{-k}, x'))/2}((2^{-k}, x')) \cup B_{\delta(y)/2}(y)$. Here

$$\mathcal{O}_j = \{(y_0, y') \in \Gamma_{10a}(x') : y_0 \in (2^{-j}, 2^{-j+1}]\}.$$

Since

$$[S_{10a}^1(u)(x')]^2 = \sum_{k=1}^{\infty} \int_{\mathcal{O}_k} |\nabla u(y)|^2 \delta(y)^{2-n} dy < \infty$$

we see that

$$\int_{\mathcal{O}_k} |\nabla u(y)|^2 \delta(y)^{2-n} dy \rightarrow 0, \quad \text{as } k \rightarrow \infty$$

and hence

$$|\tilde{u}(y) - \tilde{u}_k(x')| \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (7.3)$$

Consider now the sequence $(\tilde{u}_k(x'))_{k \in \mathbb{N}}$. We claim that it is Cauchy and hence convergent. Indeed, since $x' \in \mathcal{S}$ then $x' \in E_\lambda$ for some $\lambda > 0$ and hence

$$\sum_{k=1}^{\infty} |\tilde{u}_k(x') - \tilde{u}_{k+1}(x')| \leq \sum_{k=1}^{\infty} \frac{\sqrt{\lambda}}{2^{k/4}} < \infty.$$

From this the claim that $(\tilde{u}_k(x'))_{k \in \mathbb{N}}$ is Cauchy follows. As $\tilde{u}_k \rightarrow f$ in L^2 we therefore have $\tilde{u}_k(x') \rightarrow f(x')$ pointwise as $k \rightarrow \infty$ for all $x' \in \mathcal{S}$ (f can be modified on a set of measure zero if necessary). Combining this with (7.3) we see that

$$\tilde{u}(y) \rightarrow f(x'), \quad \text{as } y \rightarrow x' \text{ and } y \in \Gamma_a(x')$$

for all $x' \in \mathcal{S}$. This proves the lemma. \square

Lemma 7.2. *Let $1 < p < \infty$ and assume that the L^p Dirichlet problem for the operator $\mathcal{L}u = \operatorname{div}(A(x)\nabla u) + B(x) \cdot \nabla u$ is solvable on a domain $\Omega \subset \mathbb{R}^n$. Assume also that \mathcal{L} is such that the Lax-Milgram lemma applies (implying existence of the energy solutions in $W^{1,2}(\Omega; \mathbb{C})$).*

For any $f \in L^p(\partial\Omega; \mathbb{C})$ consider an approximation of f by functions $f_k \in B_{1/2}^{2,2}(\partial\Omega; \mathbb{C}) \cap L^p(\partial\Omega; \mathbb{C})$ such that

$$f_k \rightarrow f \quad \text{in } L^p(\partial\Omega; \mathbb{C}).$$

Let u_k be the energy solutions corresponding to data given by f_k . Let

$$u = \lim_{k \rightarrow \infty} u_k \quad \text{on } \Omega.$$

Then $u \in W_{loc}^{1,2}(\Omega; \mathbb{C})$ and u satisfies the estimate

$$\|\tilde{N}_{2,a}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)} \quad (7.4)$$

with $C > 0$ as in Definition 4.1. The averages

$$\tilde{u}(x) = \oint_{B_{\delta(x)/2}(x)} u(y) dy, \quad \forall x \in \Omega$$

satisfy

$$f(Q) = \lim_{y \rightarrow Q, y \in \Gamma(Q)} \tilde{u}(y), \quad \text{for a.e. } Q \in \partial\Omega. \quad (7.5)$$

We omit the proof of the lemma as it uses the same argument as in the case of real coefficients (using the estimate which follows from solvability

$$\|\tilde{N}_{2,a}(u_k - u_l)\|_{L^p(\partial\Omega)} \leq C\|f_k - f_l\|_{L^p(\partial\Omega)}$$

repeatedly). In the real case the approximating functions are chosen so that they are continuous, which (under mild assumptions (such as NTA) on regularity of $\partial\Omega$) then immediately implies

$$f_k(Q) = \lim_{y \rightarrow Q, y \in \Gamma(Q)} \tilde{u}_k(y), \quad \text{for all } Q \in \partial\Omega. \quad (7.6)$$

In our case (of complex coefficients) (7.6) is replaced by (7.1) for each u_k and f_k (the a.e. convergence is sufficient for the argument). The rest of the proof goes as in the real case giving us (7.5) for \tilde{u} and f .

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